Expressibility in the Lambda Calculus with letrec

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We investigate the relationship between finite terms in $\lambda_{\text{letrec}}$, the lambda calculus with letrec, and the infinite lambda terms they express. As there are easy examples of infinite $\lambda$-terms that, intuitively, are not unfoldings of terms in $\lambda_{\text{letrec}}$, we consider the question: How can those infinite lambda terms be characterised that are $\lambda_{\text{letrec}}$-expressible in the sense that they can be obtained as infinite unfoldings of terms in $\lambda_{\text{letrec}}$?

For ‘observing’ infinite $\lambda$-terms through repeated ‘experiments’ carried out at the head of the term we introduce two rewrite systems (with rewrite relations) $\rightarrow_{\text{reg}}$ and $\rightarrow_{\text{reg}^+}$ that decompose the term structure, and produce ‘generated subterms’ in two notions. Thereby the sort of the step can be observed as well as its target, a generated subterm. In both systems there are four sorts of decomposition steps: $\rightarrow_{\lambda}$-steps (decomposing a $\lambda$-abstraction), $\rightarrow_{\text{del}}$- and $\rightarrow_{\text{del}^*}$-steps (decomposing an application into its function and argument), and respectively, $\rightarrow_{\text{del}}$-steps (delimiting the scope of an abstraction, for $\rightarrow_{\text{reg}}$), and $\rightarrow_{S}$ (delimiting of scopes, for $\rightarrow_{\text{reg}^+}$). These steps take place on infinite $\lambda$-terms furnished with a leading prefix of abstractions for gathering previously encountered $\lambda$-abstractions and keeping the generated subterms closed. We call an infinite $\lambda$-term ‘regular’/‘strongly regular’ if its set of $\rightarrow_{\text{reg}}$-reachable/$\rightarrow_{\text{reg}^+}$-reachable generated subterms is finite. Furthermore, we analyse the binding structure of infinite $\lambda$-terms with the concept of ‘binding–capturing chain’.

Utilizing these concepts, we answer the question above by providing two characterizations of $\lambda_{\text{letrec}}$-expressibility. For all infinite $\lambda$-terms $M$, the following statements are equivalent: (i): $M$ is $\lambda_{\text{letrec}}$-expressible; (ii): $M$ is strongly regular; (iii): $M$ only has finite binding–capturing chains.

1. Introduction

A prevalent enrichment of the $\lambda$-calculus is the extension by a $\mu$ or letrec construction, the latter being a generalisation of the first. Such constructions allow for a finite representation of infinite $\lambda$-terms, which for typed $\lambda$-calculi (and thus for most functional programming

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languages) is necessary to facilitate recursion and ensure Turing completeness. A term in the \( \lambda \)-calculus with \texttt{letrec} (which we denote by \( \lambda_{\text{letrec}} \)) is usually understood as a representative for the \( \lambda \)-term obtained by infinite \texttt{letrec}-unfolding.

But it turns out that not every infinite \( \lambda \)-term can be expressed finitely in \( \lambda_{\text{letrec}} \) (for an example with an informal explanation, see Example 1.1 below). On the other hand, \( \lambda_{\text{letrec}} \)-expressible infinite \( \lambda \)-terms have infinitely many \( \lambda_{\text{letrec}} \)-representations (cf. Example 1.2). We say that a \( \lambda \)-term is ‘\( \lambda_{\text{letrec}} \)-expressible’ if it has a representation as finite term in \( \lambda_{\text{letrec}} \).

\textbf{Example 1.1 (not \( \lambda_{\text{letrec}} \)-expressible).} Consider the infinite \( \lambda \)-term of the form \( M = \lambda a.\lambda b. (\lambda c. (\lambda d. \ldots c) b) a \) with term tree as shown in Figure 1 on the right. While this term tree has a regular structure, the scopes of the abstractions in it are infinitely entangled: the scope of \( \lambda a \) reaches into the scope of \( \lambda b \), the scope of \( \lambda b \) into the one of \( \lambda c \), and so on. This feature of \( M \) can suggest the idea that it is impossible that \( M \) is the result of stepwisely ‘unrolling’ a \( \lambda_{\text{letrec}} \)-term in a manner that respects scopes. Such a process would namely ‘tile’ its result in a regular manner with finite term-context tiles having a bounded scope-nesting depth such that furthermore the scopes of abstractions contained in different context tiles do not overlap. This excludes, intuitively, the formation of the infinite entanglement of successively overlapping scopes that can be observed in \( M \). – The term \( M \) will indeed be recognised as not \( \lambda_{\text{letrec}} \)-expressible.

\textbf{Example 1.2 (\( \lambda_{\text{letrec}} \)-expressible).} The infinite \( \lambda \)-term \( N = \lambda x.x::x::\ldots \) can be expressed by \( \lambda x.\texttt{letrec} \, r = x : r \) in \( r \) as well as by \( \lambda x.\texttt{letrec} \, r = x : (x : r) \) in \( r \) (for the term graphs of these \( \lambda_{\text{letrec}} \)-terms as well as the term tree of \( N \) of their infinite unfolding, see Figure 1 on the right). The colon (:) used here is inspired from the functional programming language Haskell. It is a binary infix operator constructing a list from its arguments by putting the first argument (here: \( x \)) in front of its second argument. This example is an implementation of what is known as the \textit{repeat} function. The \( \lambda \)-calculus we deal with

\begin{center}
\begin{tikzpicture}[scale=0.8,level distance=1.5cm,level 1/.style={sibling distance=3.5cm},level 2/.style={sibling distance=2.5cm},level 3/.style={sibling distance=2cm}]

\node{\( \lambda a \)}
  child{node{\( \lambda b \)}}
  child{node{\( \lambda c \)}
    child{node{\( \lambda d \)}
      child{node{\( \vdots \)}
        child{node{\( c \)}}
      }
      child{node{\( b \)}}
    }
    child{node{\( a \)}}
  }
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}[scale=0.8,level distance=1.5cm,level 1/.style={sibling distance=3.5cm},level 2/.style={sibling distance=2.5cm},level 3/.style={sibling distance=2cm}]

\node{\( \lambda a \)}
  child{node{\( \lambda b \)}}
  child{node{\( \lambda c \)}
    child{node{\( \lambda d \)}
      child{node{\( \vdots \)}
        child{node{\( c \)}}
      }
      child{node{\( b \)}}
    }
    child{node{\( a \)}}
  }
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}[scale=0.8,level distance=1.5cm,level 1/.style={sibling distance=3.5cm},level 2/.style={sibling distance=2.5cm},level 3/.style={sibling distance=2cm}]

\node{\( \lambda x \)}
  child{node{\( \vdots \)}
    child{node{\( x \)}}
  }
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}[scale=0.8,level distance=1.5cm,level 1/.style={sibling distance=3.5cm},level 2/.style={sibling distance=2.5cm},level 3/.style={sibling distance=2cm}]

\node{\( \lambda x \)}
  child{node{\( \vdots \)}
    child{node{\( x \)}}
  }
\end{tikzpicture}
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\node{\( \lambda x \)}
  child{node{\( \vdots \)}
    child{node{\( x \)}}
  }
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}[scale=0.8,level distance=1.5cm,level 1/.style={sibling distance=3.5cm},level 2/.style={sibling distance=2.5cm},level 3/.style={sibling distance=2cm}]

\node{\( \lambda x \)}
  child{node{\( \vdots \)}
    child{node{\( x \)}}
  }
\end{tikzpicture}
\end{center}

Fig. 1. Termgraphs for Example 1.1 and Example 1.2
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here does not explicitly include operators. Therefore in the context of this work the colon can be viewed as a free variable or a closed $\lambda$-term.

Note that, if colon is viewed as a closed $\lambda$-term that contains $\lambda$-abstractions, then, contrary to the term $M$ from Example 1.1, abstraction scopes in the term tree of $N$ are organised in blocks of of bounded scope-nesting depth (for example each occurrence of colon then is such a block), and no infinite entanglement of scopes takes place.

**Overview of the paper.** Section 2 is concerned with terminology and notation used for known formalisms. In Section 3 a rewrite system for unfolding terms in the $\lambda$-calculus with letrec is formulated. In Section 4 we introduce rewriting systems for decomposing infinite $\lambda$-terms into ‘generated subterms’, and we show some properties of these systems in connection to so-called scope-delimiting strategies. Also in this section, we define regularity and strong regularity for infinite $\lambda$-terms employing the concepts os generated subterms and scope-delimiting strategies. In Section 5 we adapt the rewrite systems for decomposing infinite $\lambda$-terms and the notions of scope-delimiting strategies to the $\lambda$-calculus with letrec. In Section 6, we develop proof systems that are sound and complete for the notions of regularity and strong regularity, for equality of strongly regular infinite $\lambda$-terms, and for the property of a $\lambda$letrec-term to unfold to an infinite $\lambda$-term. In Section 7 we examine the binding structure of infinite $\lambda$-terms (binding–capturing chains) and connect to the concepts introduced so far. In Section 8 we establish the correspondence between strong regularity and $\lambda$letrec-expressibility for infinite $\lambda$-terms. In Section 9 we introduce ‘$\lambda$-transition graphs’ of infinite $\lambda$-terms and of $\lambda$letrec-terms as labelled transition graphs in which the edges carry one of the four different labels 0, 1, $\lambda$, and $S$. In Section 10 we summarize and provide an outlook on possible results that are related, and on potential applications of the concepts we introduce.

2. Preliminaries

In this section we gather most of the basic known concepts that play a vital part in the rest of our paper. Some central notions concerning rewriting are recapitulated from (Terese, 2003), while for others we provide references. Some variations of known concepts that are tailor made for our purposes are formulated in definition environments.

We let $N = \{0,1,2,\ldots\}$ and $N^{*} = N \setminus \{0\}$. For a partial function $f : A \rightarrow B$, and $a \in A$ we denote by $f(a)^{\downarrow}$ the fact that $f$ is defined for $a$, and by $f(a)^{\uparrow}$ that $f$ is not defined for $a$. The domain of $f$ is the set $\text{dom}(f) := \{a \in A \mid f(a)^{\downarrow}\} \subseteq A$, and the range of $f$ the set $\text{ran}(f) := \{f(a) \mid a \in A, f(a)^{\downarrow}\} \subseteq B$.

For relations $R \subseteq A \times B$ and $S \subseteq B \times C$ we denote by $R \cdot S$ the composition of $R$ with $S$ defined by $R \cdot S := \{(x,z) \mid (\exists y \in B)(x,y) \in R \land (y,z) \in S\} \subseteq A \times C$, and by $R^{\ast}$ the reflexive and transitive closure of $R$ under composition, which is defined by $R^{\ast} := \bigcup_{i \in \mathbb{N}} R^{i}$ where $R^{0} := \text{id}_{A} := \{(x,x) \mid x \in A\}$ and, for all $i \in \mathbb{N}$, $R^{i+1} := R \cdot R^{i}$.

**Abstract Rewriting Systems.** An abstract rewriting system (ARS) is a quadruple $(A, \Phi, \text{src}, \text{tgt})$ consisting of a set $A$ of objects, a set $\Phi$ of steps, and $\text{src}, \text{tgt} : \Phi \rightarrow A$, the *source* and *target* functions. We will always assume that $A \cap \Phi = \emptyset$. For objects $a \in A$ we denote by $\Phi_{\text{out}}(a)$ and by $\Phi_{\text{in}}(a)$ the set of steps in $\Phi$ that depart (are outgoing steps)
from a, and that arrive (are incoming steps) at a, respectively. We say that an ARS is finite if its set of steps is finite. For ARSs \( A_i = (A_i, \Phi_i, \text{src}_i, \text{tgt}_i) \) for \( i \in \{1, 2\} \) we say that \( A_1 \) is a sub-ARS of \( A_2 \) if \( A_1 \subseteq A_2 \), \( \Phi_1 \subseteq \Phi_2 \), and \( \text{src}_1, \text{tgt}_1 \) are the restrictions of \( \text{src}_2 \) and \( \text{tgt}_2 \), respectively, to \( \Phi_1 \), which are required to be total functions (this implies that, for all \( \phi \in \Phi_1 \), it holds that \( \text{src}_2(\phi) = \text{src}_1(\phi) \in A_1 \), and \( \text{tgt}_2(\phi) = \text{tgt}_1(\phi) \in A_1 \).

**Induced sub-ARS.** For an object \( a \in A \) of an ARS \( A = (A, \Phi, \text{src}, \text{tgt}) \) we denote by \( (a \rightarrow) : (A', \Phi', \text{src}', \text{tgt}') \) the sub-ARS of \( A \) induced by \( a \), where \( A' \) comprises only the objects from \( A \) that are reachable from \( a \) by an arbitrary number of steps (or no steps) and with \( \Phi', \text{src}', \text{tgt}' \) being the restrictions of \( \Phi, \text{src}, \text{tgt} \) to the objects in \( A' \) and to steps between objects of \( A' \).

**Bisimulations between ARSs.** Let \( A_i = (A_i, \Phi_i, \text{src}_i, \text{tgt}_i) \) for \( i \in \{1, 2\} \) be ARSs. A relation \( B \subseteq (A_1 \times A_2) \cup (\Phi_1 \times \Phi_2) \), which relates objects with objects and steps with steps, is called an ARS-bisimulation if:

- if \( a_1 B a_2 \), then \( B \) relates each step from \( a_1 \) to some step from \( a_2 \) (forth condition), and each step from \( a_2 \) to some step from \( a_1 \) (back condition);
- if \( \phi_1 B \phi_2 \) with \( \phi_1 : a_1 \rightarrow a_1' \) and \( \phi_2 : a_2 \rightarrow a_2' \), then \( a_1 B a_2 \) and \( a_1' B a_2' \).

**Labellings of ARSs.** Let \( A = (A, \Phi, \text{src}, \text{tgt}) \) and \( A' = (A', \Phi', \text{src}', \text{tgt}') \) be ARSs.

(i) An ARS-bisimulation \( L \) between \( A \) and \( A' \) is called a labelling of \( A \) to \( A' \), and \( A' \) the L-labelled version of \( A \), if the converse \( L^{-1} \) of \( L \) is a function \( L^{-1} : A' \cup \Phi' \rightarrow A \cup \Phi \), and if additionally, for all \( a' \in A' \) and \( a \in A \) with an \( L \) \( a' \), the restriction \( L^{-1}|_{\Phi'_{\text{out}}(a')} : \Phi'_{\text{out}}(a') \rightarrow \Phi_{\text{out}}(a) \) of \( L^{-1} \) to the steps departing from \( a' \) is bijective.

(ii) A rewrite labelling \( L \) of \( A \) to \( A' \) is a pair \( (L, l) \) consisting of a labelling \( L \) of \( A \) to \( A' \) together with an initial labelling function \( l \) mapping objects of \( A \) to bisimilar objects of \( A' \).

**Strategies.** A history-free strategy for an abstract rewriting system \( A \) is a sub-ARS of \( A \) that has the same objects, and the same normal forms as \( A \). A history-aware strategy for an abstract rewriting system \( A \) is a history-free strategy for the L-labelled version of \( A \) with respect to, and together with, a rewrite labelling \( (L, l) \) of \( A \). By a strategy for \( A \) we will mean a history-free strategy or a history-aware strategy for \( A \).

**Remark.** Let \( S \) be a history-aware strategy of \( A \), and let \( A' \) be that L-labelled version of \( A \) which \( S \) is a history-free strategy of. Then \( S \) projects to a history-free strategy \( \tilde{S} \) of \( A \). The projection is defined by \( \tilde{L} \), which induces a local bijective correspondence of outgoing steps of related sources of \( A \) and \( A' \). Mind that for deterministic \( S \), \( \tilde{S} \) may become non-deterministic. Furthermore, every rewrite sequence according to \( S \) in \( A' \) projects to a unique rewrite sequence in \( A \) (which is a rewrite sequence according to \( \tilde{S} \)).

The last mentioned fact makes it possible to speak, for a given rewrite labelling, of rewrite sequences of a history-aware strategy on the objects of the original ARS. Let \( S \) be a history-aware strategy for an ARS \( A \), and \( a \) an object of \( A \). Suppose that \( S \) is a sub-ARS of the \( L \)-labelled version \( A' \) of \( A \) for some rewrite labelling \( (L, l) \) of \( A \). Then by a rewrite sequence of \( S \) on \( a \) (in \( A \)) we will mean the projection to \( A \) of a rewrite sequence of \( S \) (in \( A' \)) on the result \( l(a) \) of the initial labelling applied to \( a \).
Rewrite relations: notation and properties. For the single-step rewrite relation induced by an ARS we use the arrow symbol \( \rightarrow \) possibly subscripted with appropriate names. Let \( \rightarrow \subseteq A \times A \) be a rewrite relation. We denote by \( \rightarrow^* \) the many-step rewrite relation induced by \( \rightarrow \), by which we mean the reflexive and transitive closure of \( \rightarrow \). By \( \rightarrow^+ \) we denote the one-or-more-step rewrite relation of \( \rightarrow \), the transitive closure of \( \rightarrow \). By \( \rightarrow^- \) we mean the zero-or-one-step rewrite relation of \( \rightarrow \), the reflexive closure of \( \rightarrow \). By a normal form of \( \rightarrow \) we mean an \( a \in A \) such that there is no \( a' \in A \) with \( a \rightarrow a' \). By \( \rightarrow^1 \) we mean the reduction to normal form rewrite relation induced by \( \rightarrow \), the restriction of \( \rightarrow \) to a relation with the normal forms of \( \rightarrow \) as codomain: \( \rightarrow^1 = \{(a, a') \mid a \rightarrow a', a' \text{ is normal form of } \rightarrow \} \).

The rewriting properties below are reformulations, and some are slight variations, of known properties of rewrite relations.

**Definition 2.1.** Let \( \rightarrow_1, \rightarrow_2, \rightarrow_3 \) be rewrite relations. The rewrite relation \( \rightarrow_1 \) is called cofinal for \( \rightarrow_2 \) if \( \rightarrow_2 \subseteq \rightarrow_1 \cdot \rightarrow_2 \). We say that \( \rightarrow_1 \) is cofinal for \( \rightarrow_2 \) with trailing \( \rightarrow_3 \)-steps if \( \rightarrow_2 \subseteq \rightarrow_1 \cdot \rightarrow_2 \cdot \rightarrow_3 \). Furthermore we say that \( \rightarrow_1 \) factors into \( \rightarrow_2 \) and \( \rightarrow_3 \) if \( \rightarrow_1 \subseteq \rightarrow_2 \cdot \rightarrow_3 \).

We will several times use the following specific version of König’s Lemma.

**König’s Lemma.** Let \( G = (V, E) \) be an undirected graph with set \( V \) of vertices and set \( E \) of edges. Suppose that \( G \) has infinitely many vertices (\( V \) is infinite), that it is connected (for all vertices \( v, w \in V \) there exists a path in \( G \) from \( v \) to \( w \)) and that every vertex has finite degree (it is adjacent to only finitely many other vertices in \( G \)). Then for every vertex \( v \in V \), \( G \) contains an infinitely long simple path from \( v \), that is, a path starting at \( v \) without repetition of vertices.\(^1\)

**Combinatory Reduction Systems.** Many of the formalisations we introduce are based on the framework of Combinatory Reduction Systems (CRSs) (Klop, 1980), (Klop et al., 1993) (Terese, 2003, Sec. 11.3), and, in particular, on infinitary Combinatory Reduction Systems (iCRSs) (Ketema and Simonsen, 2011).

**Infinite CRSs.** When speaking of ‘infinite terms’ for CRSs over some signature we draw on (Terese, 2003, 12.4) and (Ketema and Simonsen, 2011) where meta-terms of iCRSs are defined by means of metric completion. The metric is defined on \( \alpha \)-equivalence classes of finite preterms dependent on the minimal depth at which two finite preterms belonging to the equivalence classes have a ‘conflict’. The objects formed by the metric completion process can be represented as equivalence classes of infinite preterms, we call them iCRS-preterms, with respect to a notion of \( \alpha \)-equivalence that again is based on the notion of ‘conflict’ (see also (Terese, 2003, Def. 12.4.1)). Hereby iCRS-preterms are infinite ordered dyadic trees in which each node is either labelled by a variable name, and then the node does not have a successor, or by named abstractions \( \lambda x \) (with some

\(^1\) This formulation corresponds to the following original formulation by Dénes König on page 80 in (König, 2001): “Satz 3: Jeder unendliche zusammenhängende Graph \( G \) endlichen Grades besitzt einen einseitig unendlichen Weg, wobei der Anfangspunkt \( P_0 \) dieses Weges beliebig vorgeschrieben werden kann.” This in connection with the definition on page 10 in (König, 2001): “Eine unendliche Menge von Kanten \( P_iP_{i+1} \) (\( i = 0, 1, 2, \ldots \) in mf.), bzw. der durch sie gebildete Graph, heißt ein einseitig unendlicher Weg, falls für \( i \neq j \) stets \( P_i \neq P_j \) ist.”
Fig. 2. Schroer-style proof system $A^\infty(\Sigma)$ for $\alpha$-equivalence of iCRS-preterms over
signature $\Sigma$: for every $f \in \Sigma$ with arity $n$, $A^\infty(\Sigma)$ contains a rule $f$. In instances of the rule
$[\cdot]$, the constant $c$ is chosen fresh for $s$ and $t$. Substitution which occurs in the assumption
of $[\cdot]$ denotes substitution by variable replacement on iCRS-preterms. It needs not to be
capture-avoiding because of the freshness of the substituant.

variable name $x$), and then the node has a single successor node, or by an abstraction
symbol, and then the node has a right and a left successor node.

For denoting infinite preterms (and later terms) we use, as much as possible, usual
notation for dealing with finite terms. A slight exception is our use, in certain situations,
of a finite-CRS-based notation for infinite $\lambda$-terms that are not $\lambda_{\text{letrec}}$-expressible (e.g. see Example 4.25).

By iCRS-terms we will mean $\alpha$-equivalence classes of iCRS-preterms. The notion of
$\alpha$-equivalence on iCRS-preterms based on the absence of conflicts can be described by
provability in the proof system $A^\infty(\Sigma)$ in Figure 2 which is a variant of a proof system
due to Schroer (see (Hendriks and van Oostrom, 2003)).

**Definition 2.2 (\(\alpha\)-equivalence for iCRS preterms, Schroer-style proof system).**
The proof system $A^\infty(\Sigma)$ for $\alpha$-equivalence on iCRS-preterms over signature $\Sigma$ consists of
the axioms and rules displayed in Figure 2, each of which contains a rule $f$ for every $f \in \Sigma$.
Provability in $A^\infty(\Sigma)$ of an equation between preterms is defined as the existence of a
possibly infinite, completed derivation: for example, by $A^\infty\vdash s = t$ we mean the existence
of a possibly infinite proof tree $D^\infty$ with conclusion $s = t$ such that maximal threads from
the conclusion upwards either have length $\omega$, or have finite length and end at a leaf that
carries an axiom $0$. (We will generally use the decorated turnstyle symbol $\vdash$ to indicate
provability by completed, possibly infinite derivations.)

However, closer to coinductive proof systems for infinite $\lambda$-terms that we develop
is the following different, but equivalent characterisation of $\alpha$-equivalence for infinite
iCRS-preterms, a variant for iCRS-terms of a proof system for $\alpha$-equivalence between
finite $\lambda$-terms due to Kahrs (see (Hendriks and van Oostrom, 2003)).

**Definition 2.3 (\(\alpha\)-equivalence for iCRS preterms, Kahrs-style proof system).**
The proof system $A^\infty(\Sigma)$ for $\alpha$-equivalence on iCRS-preterms over signature $\Sigma$ consists
of the axioms and the rules in Figure 3 with, for every $f \in \Sigma$, a rule $f$. Provability of an
equation between preterms in $A^\infty(\Sigma)$ is defined, analogously as in 2, as the existence of a
possibly infinite, completed derivation.

This formulation of $\alpha$-equivalence for infinite $\lambda$-terms will be the key to our formulation
of a ‘coinduction principle’ for infinite $\lambda$-terms in Theorem 9.7.

**Infinite rewrite relation.** For an iCRS with rewrite relation $\rightarrow$, we denote by $\Rightarrow$
the infinitary rewrite relation induced by strongly convergent and continuous rewrite
sequences of arbitrary (countable) ordinal length. Hereby strong convergence means that, at every limit ordinal, the rewrite activity in the terms of the rewrite sequence tends to infinity. Continuity means that the terms of the rewrite sequence converge, in the metric space of infinite terms, at every limit ordinal. By \( \rightarrow^\omega \) we will denote the rewrite relation induced by strongly continuous \( \rightarrow \)-rewrite sequences of length \( \omega \).

**Labelled transition systems, labelled transition graphs.** A *labelled transition system (LTS)* is a triple \( \mathcal{L} = (S, \rightarrow) \) consisting of a set \( S \) of states, a set \( A \) of labels, and a set \( \rightarrow \subseteq S \times A \times S \) of \( A \)-labelled transitions. Labelled transitions \( (s_1, a, s_2) \in S \times A \times S \) will be indicated as \( s_1 \rightarrow^a s_2 \).

A *labelled transition graph (LTG)* \( G \) is a pointed LTS, that is, \( G = (S, A, x, \rightarrow) \) where \((S, A, \rightarrow)\) is an LTS, and \( x \in S \), which is called the *initial state*.

**Bisimulation between LTSs, LTGs.** Let \( \mathcal{L}_1 = (S_1, A, \rightarrow_1) \) and \( \mathcal{L}_2 = (S_2, A, \rightarrow_2) \) be a LTSs over a common set of labels. A *bisimulation on \( \mathcal{L} \)* is a binary relation \( R \subseteq S_1 \times S_2 \) that satisfies, for all \( s \in S_1 \) and \( t \in S_2 \):

(i) if \( s \rightarrow_1 t \) and \( s \rightarrow^a_1 s' \), then there exists \( t' \in S_2 \) such that \( t \rightarrow^a_2 t' \) and \( s' \rightarrow_1 t' \);
(ii) if \( s \rightarrow_1 t \) and \( t \rightarrow^a_2 t' \), then there exists \( s' \in S_1 \) such that \( s \rightarrow^a_1 s' \) and \( s' \rightarrow_2 t' \).

Two states \( s \in S_1 \) and \( t \in S_2 \) are *bisimilar*, denoted by \( s \equiv t \), if there exists a bisimulation \( R \) such that \( s \rightarrow R t \).

Two LTGs \( G_1 = (S_1, A, x_1, \rightarrow_1) \) and \( G_2 = (S_2, A, x_2, \rightarrow_2) \) are *bisimilar* if there is a bisimulation on the underlying LTSs that relates the initial state \( x_1 \) of \( G_1 \) with the initial state \( x_2 \) of \( G_2 \).

### 3. The \( \lambda \)-calculus and the \( \lambda_{\text{letrec}} \)-calculus

This section provides the definitions for terms in the \( \lambda \)-calculus and the \( \lambda_{\text{letrec}} \)-calculus that we will be using throughout this work. We define CRS signatures and a rewriting system \( R_{\xi} \) for unfolding \( \lambda_{\text{letrec}} \)-terms to obtain infinite \( \lambda \)-terms.

**Definition 3.1 (first-order representation of \( \lambda \) and \( \lambda_{\text{letrec}} \)).** Let \( \mathcal{X} = \{ x_0, x_1, x_2, \ldots \} \) be a set of variable names for \( \lambda \)-abstractions, and \( \mathcal{R} \) be a set of names for recursion variables. We will use \( x, y, z \) as syntactical variables for variable names bound by \( \lambda \)-abstraction, and \( f, g, h \) for recursion variable names; and similarly, we will use \( L, P, Q \)
for terms. The set of $\lambda_{letrec}$-terms is inductively defined by the following grammar:

\[
\begin{align*}
\text{(term)} \quad L &::= \lambda x. L \\
&\quad | \quad LL \\
&\quad | \quad x \\
&\quad | \quad \text{letrec } B \text{ in } L \\
\text{(binding group)} \quad B &::= f_1 = L \ldots f_n = L \\
&\quad (f_1, \ldots, f_n \in R \text{ all distinct})
\end{align*}
\]

The set of $\lambda$-terms is defined by a reduced form of the grammar with the letrec alternative and the binding group rule left out.

On this grammar the following rules describe unfolding of $\lambda_{letrec}$-terms in an informal manner:

\[
\begin{align*}
(\varrho^\text{a}_\text{letrec}) : \quad &\text{letrec } B \text{ in } L_0 \text{ } L_1 \rightarrow (\text{letrec } B \text{ in } L_0 ) (\text{letrec } B \text{ in } L_1) \\
(\varrho^\text{b}_\text{letrec}) : \quad &\text{letrec } B \text{ in } \lambda x. L_0 \rightarrow \lambda x. \text{letrec } B \text{ in } L_0 \\
(\varrho^\text{rec}) : \quad &\text{letrec } B_0 \text{ in } \text{letrec } B_1 \text{ in } L \rightarrow \text{letrec } B_0, B_1 \text{ in } L \\
(\varrho^\text{red}) : \quad &\text{letrec } f_1 = L_1 \ldots f_n = L_n \text{ in } L \rightarrow \text{letrec } f_{j_1} = L_{j_1}, \ldots, f_{j_n} = L_{j_n} \text{ in } L \\
&(\text{if } f_{j_1}, \ldots, f_{j_n} \text{ are the recursion variables reachable from } L) \\
(\varrho^\text{nil}) : \quad &\text{letrec } \text{ in } L \rightarrow L
\end{align*}
\]

The names of the first four rules are chosen to reflect the kind of term that resides inside of the in-part of the letrec-term, which helps to see that the rules are complete in the sense that every term of the form letrec $B$ in $L$ is a redex.

We will use higher-order notation and rules to reason about the $\lambda$-calculus and the $\lambda$-calculus with letrec, which immediately validates our results for $\alpha$-equivalence classes instead of just preterms.

**Remark 3.2 (infinitary rewriting).** We use CRSs as a rewriting framework since until now infinitary rewriting theory has only been developed for CRSs yet (Ketema and Simonsen, 2009; Ketema and Simonsen, 2010; Ketema and Simonsen, 2011).

For formulating the above rules as a CRS, we provide CRS-signatures for $\lambda$ and $\lambda_{letrec}$.

**Definition 3.3 (CRS signatures for $\lambda$ and $\lambda_{letrec}$).** The CRS-signature for $\lambda$ consists of the set $\Sigma_\lambda = \{ \text{app, abs} \}$ where app is a unary and abs a binary function symbol. The CRS-signature $\Sigma_{\lambda_{letrec}}$ consists of the countably infinite set $\Sigma_{\lambda_{letrec}} = \Sigma_\lambda \cup \{ \text{let}_n, \text{rec-in}_n \mid n \in \mathbb{N} \}$ of function symbols, where, for $n \in \mathbb{N}$, the symbols let$_n$ and rec-in$_n$ have arity $n$.

By $\text{Ter}(\lambda)$ and $\text{Ter}(\lambda_{letrec})$ we denote the set of closed CRS-terms over $\Sigma_\lambda$ and $\Sigma_{\lambda_{letrec}}$ respectively, with the restriction that

- the symbols let$_n$ and rec-in$_n$ only occur as patterns of the form
  \[
  \text{let}_n([f_1 \ldots f_n] \text{rec-in}_n(M_1, \ldots, M_n, M)) \quad \text{for some terms } M_1, \ldots, M_n, M \in \text{Ter}(\lambda)
  \]
- and that otherwise a CRS abstraction can only occur directly beneath an abs-symbol.
The formulation here of the rule scheme $\varrho$ Ter will later be specified more formally in Definition 6.1 and Definition 6.2.

We will use $M$, $N$, $O$ used as syntactical variables for terms in $Ter(\lambda)$. And by $Ter(\Lambda_{\text{letrec}})$ we denote the set of CRS-terms over $\Sigma$, for which we will use the symbols $L$, $P$, $Q$ as syntactical variables.

**Example 3.4.** The term in Example 1.1 in CRS notation:

$$\text{abs}([a] \text{abs}([b] \text{app}([c] \text{app}(\text{abs}([d] \text{app}(\ldots, c)), b)), a))$$

**Example 3.5.** The terms in Example 1.2 in CRS notation:

$$\text{abs}([x] \text{let}_{1}([r] \text{rec-in}_{1}(\text{app}(\ldots, r))), r)$$

$$\text{abs}([x] \text{let}_{1}([r] \text{rec-in}_{1}(\text{app}(\ldots, r)), r))$$

**Definition 3.6 (terms in $\Lambda^\omega$).** We denote by $Ter((\Lambda^\omega))$ the set of finite and infinite CRS-terms over the signature $\Sigma$. Note that the set of infinite $\lambda$-terms subsume finite $\lambda$-terms, thus whenever we speak of an infinite $\lambda$-term in fact we refer to a potentially infinite $\lambda$-term.

**Definition 3.7 (CRS for unfolding in $\Lambda_{\text{letrec}}$).** The CRS $R_\varrho$ for unfolding $\Lambda_{\text{letrec}}$-terms is the CRS for terms over the signature $\Sigma_{\text{letrec}}$ (see Definition 3.3) with the following rules in which $n$ varies among numbers in $\mathbb{N}^+$:

$$(\varrho^\Lambda_{\text{letrec}}) : \begin{array}{l}
\text{let}_{n}([f]\text{rec-in}_{n}(X_{1}(\bar{f}), \ldots, X_{n}(\bar{f}), \text{app}(Z_{0}(\bar{f}), Z_{1}(\bar{f})))) \\
\rightarrow \text{app}((\text{let}_{n}([f]\text{rec-in}_{n}(\ldots, X_{n}(\bar{f}), Z_{0}(\bar{f})))), (\text{let}_{n}([f]\text{rec-in}_{n}(\ldots, X_{n}(\bar{f}), Z_{1}(\bar{f}))))
\end{array}$$

$$(\varrho^\Lambda_{\text{red}}) : \begin{array}{l}
\text{abs}([x] \text{let}_{n}([f]\text{rec-in}_{n}(X_{1}(\bar{f}), \ldots, X_{n}(\bar{f}), \text{abs}([x] Z(\bar{f}, x)))) \\
\rightarrow \text{abs}([x] \text{let}_{n}([f]\text{rec-in}_{n}(X_{1}(\bar{f}), \ldots, X_{n}(\bar{f}), Z(\bar{f}, x))))
\end{array}$$

$$(\varrho^\Lambda_{\text{rec}}) : \begin{array}{l}
\text{let}_{n}([f]\text{rec-in}_{n}(X_{1}(\bar{f}), \ldots, X_{n}(\bar{f})), \text{let}_{m}([g]\text{rec-in}_{m}(Y_{1}(\bar{f}, \bar{g}), \ldots, Y_{n}(\bar{f}, \bar{g}) Z(\bar{f}, \bar{g})))) \\
\rightarrow \text{let}_{n+m}([f]\text{rec-in}_{n+m}(X_{1}(\bar{f}), \ldots, X_{n}(\bar{f}), Y_{1}(\bar{f}, \bar{g}), \ldots, Y_{m}(\bar{f}, \bar{g}) Z(\bar{f}, \bar{g})))
\end{array}$$

$$(\varrho^\Lambda_{\text{nii}}) : \text{let}_{0}(\text{rec-in}_{0}(Z)) \rightarrow Z$$

$$(\varrho^\Lambda_{\text{red}}) : \begin{array}{l}
\text{let}_{n}([f_{1} \ldots f_{n}] \text{rec-in}_{n}(X_{1}(\bar{f}_{1}, \ldots, f_{m_{1}+1}), \ldots, X_{n}(f_{1,n}, \ldots, f_{m_{n}+1,n})), Z(f_{1,n+1}, \ldots, f_{m_{n+1}+1,n+1}))) \\
\rightarrow \text{let}_{n}([f_{1} \ldots f_{j_{n}}] \text{rec-in}_{n}(X_{1}(\bar{f}_{1}, \ldots, f_{m_{1}+1}), \ldots, X_{j_{n}}(L_{1,j_{1}+1}, \ldots, L_{m_{j_{1}}+1,j_{1}}), \ldots, X_{j_{n}}(L_{1,j_{n}+1}, \ldots, L_{m_{j_{n}}+1,j_{n}}), Z(f_{1,n+1}, \ldots, f_{m_{n+1}+1,n+1})))
\end{array}$$

The formulation here of the rule scheme $\varrho^\Lambda_{\text{red}}$ for reducing the binding group of a letrec-binding by removing unreachable equations needs additional explanation. On the right-hand side, the terms denoted by the symbol $L$ with different indices are defined as
Fig. 4. Unbounded growth of the binding group in a $\rightarrow_\varphi$-rewrite sequence on a $\lambda_{\text{letrec}}$-term that does not contain $\rightarrow_\varphi,\text{red}$-steps.

follows:

$$L_{y,j_x} := \begin{cases} f_{y,x} & \text{if } i_{y,x} \in \{j_1, \ldots, j_n\} \\ \lambda w.w & \text{else} \end{cases}$$

(for all $1 \leq x \leq n'$ and $1 \leq y \leq m_{j_x}$),

relative to the indices $j_1, \ldots, j_{n'}$ with $1 \leq j_1 < \ldots < j_{n'} \leq n$ that are precisely the indices that are reachable from the indices in $\{i_{1,n+1}, \ldots, i_{m_{n+1},n+1}\}$ (the indices $l$ of $f_l$ that are applied to the variable $X$) via the binary reachability relation $\Rightarrow$ on $\{1, \ldots, n\}$ that is defined, for all $x, y \in \{1, \ldots, n\}$, by: $x \Rightarrow y$ if and only if there exist $a z \in \{1, \ldots, m_x\}$ with $i_{z,x} = y$, formally that is: $\{j_1, \ldots, j_{n'}\} = \{l \mid i_{y,x} \Rightarrow l \text{ for some } i \in \{i_{1,n+1}, \ldots, i_{m_{n+1},n+1}\}\}$.

The ARS induced by the CRS $R_\varphi$ will be denoted by $R_\varphi$. And by $\rightarrow_\varphi,\varnothing$, $\rightarrow_\varphi,\lambda$, $\rightarrow_\varphi,\text{nil}$, $\rightarrow_\varphi,\text{rec}$, $\rightarrow_\varphi,\text{letrec}$, and $\rightarrow_\varphi,\text{red}$ we denote the rewrite relations of both $R_\varphi$ and $R_\varphi$ that are induced by the rules $g_\varphi^0$, $g_\varphi^\lambda$, $g_\varphi^{\text{nil}}$, $g_\varphi^{\text{rec}}$, $g_\varphi^{\text{letrec}}$, and $g_\varphi^{\text{red}}$, respectively.

Remark 3.8 (Motivation of the rules $g_\varphi^{\text{red}}$ and $g_\varphi^{\text{nil}}$ in $R_\varphi$). The purpose of taking up the rule $g_\varphi^{\text{red}}$, together with the rule $g_\varphi^{\text{nil}}$ into the CRS $R_\varphi$ consists in preventing unbounded growth of binding groups during unfolding. Consider for instance the outermost rewrite sequence on the term $\text{letrec } f = \text{letrec } g = f g \text{ in } g$ shown in Figure 4. Applications of the rule $g_\varphi^{\text{red}}$ are able to remove unreachable equations in binding groups.

While restricting the size of binding groups during unfolding is a sensible constraint on the unfolding process, it is not strictly necessary to define the unfolding of a $\lambda_{\text{letrec}}$-term. We could also use a rule $g_\varphi^{\text{free}}$ in place of $g_\varphi^{\text{red}}$ and $g_\varphi^{\text{nil}}$ where $g_\varphi^{\text{free}}$ is (informally) defined as:

$$(g_\varphi^{\text{free}}) : \text{letrec } f_1 = L_1 \ldots f_n = L_n \text{ in } L \rightarrow L$$  (if $f_1, \ldots, f_n$ do not occur in $L$)

which allows steps like $\lambda x. \text{letrec } B$ in $x \rightarrow \lambda x.x$, and thus allows to move bound variables out of the in-part of letrec-expressions. (Note that in $R_\varphi$ such a step can be simulated by
Expressibility in the Lambda Calculus with \textit{letrec}

\[ \rightarrow_{\text{red}} \] step followed by a \( \rightarrow_{\text{nil}} \) step.) We will, however, embed the unfolding rules into other rewriting systems of which we wish to perform unfolding in a lazy way such that the number of derivable terms is bounded. The approach with the \( \varrho^\text{free} \)-rule runs counter to that idea, as it easily leads to an unbounded growth of binding groups.

\textbf{Remark 3.9 (shape of the rule \( \varrho^\text{red} \) in \( R^\varrho \)).} The rewrite rules \( \varrho^\text{red} \) of \( R^\varrho \) are not ‘fully extended’ as the metavariables \( X_i \) occurring in the left-hand side of the rule do not have to be instantiated with all recursion variables \( f_1, \ldots, f_n \) bound in the abstraction prefix. This is due to the design of this rule scheme in which reachability of a recursion variable \( f_j \) from the in-part of the formalised \textit{letrec}-term is defined by extracting from the format of the specific instance which recursion variables \( f_1, \ldots, f_n \) occur in which of the metavariables \( X_1, \ldots, X_n \).

In applications of a rule \( \varrho^\text{red} \), unreachable recursion variables are removed from the abstraction prefix on the right hand side. Since the format of CRSs requires that corresponding metavariables on the left- and on the right-hand side of a rule must have the same arity, occurrences of unreachable recursion variables as arguments for metavariables describing the binding group of a reachable recursion variable cannot simply disappear on the right-hand side. In the definition above, a specific, but arbitrarily chosen \( \lambda \)-term, namely \( \lambda w.w \), is substituted for such occurrences of unreachable recursion variables. This does not interfere with the unfolding operation defined later.

Furthermore we profit from the property of normal forms w.r.t. \( \varrho^\text{red} \) that the set of free variables of the \( \lambda \textit{letrec} \)-term corresponds to the set of free variables of its unfolding, which we will utilise in the mention rewriting systems later on.

\textbf{Definition 3.10 (reduced \( \lambda \textit{letrec} \)-terms).} A \( \lambda \textit{letrec} \)-term \( L \) is called \emph{reduced} if it is a normal form with respect to \( \rightarrow_{\varrho, \text{nil}} \) and \( \rightarrow_{\varrho, \text{red}} \). If a \( \lambda \textit{letrec} \)-terms \( P \) reduces to a reduced term \( L \) by exclusively \( \rightarrow_{\varrho, \text{nil}} \) - and \( \rightarrow_{\varrho, \text{red}} \)-steps, then \( P \) is called a \emph{reduced form} of \( L \).

\textbf{Proposition 3.11 (confluence of \( \textit{letrec} \)-unfolding).} \( R^\varrho \) is confluent.

\textit{Proof.} We give a proof based on decreasing diagrams (Terese, 2003, Sec. 14.2). Most pairings \(( \varrho^\text{\varrho, nil}, \varrho^\text{\varrho, red} \)) of \( R^\varrho \)’s rewriting rules are unproblematic as they have an elementary diagram of the form

\[ \leftarrow_a \cdot \rightarrow_b \subseteq \rightarrow_a \cdot \leftarrow_b \]

which does not impose any restriction on the order required for the argument. One (also unproblematic) exception is the pairing \(( \varrho^\text{\varrho, nil}, \varrho^\text{\varrho, letrec} \)) which in one special case commutes only together with \( \varrho^\text{\varrho, red} \) in a diagram of the form:

\[ \leftarrow_{\varrho, \text{nil}} \cdot \rightarrow_{\varrho, \text{letrec}} \subseteq \rightarrow_{\varrho, \text{red}} \]

The rules \( \varrho^\text{\varrho, nil} \) and \( \varrho^\text{\varrho, letrec} \) deserve closer attention since they are duplicating rules. They commute with any other rule \( \varrho^\text{\varrho, red} \) in a manner such they induce \( \varrho^\text{\varrho, nil} > \varrho^\text{\varrho, red} \) or \( \varrho^\text{\varrho, letrec} > \varrho^\text{\varrho, red} \) respectively. The pairing of the rules \( \varrho^\text{\varrho, red} \) and \( \varrho^\text{\varrho, nil} \) can be resolved by indexing them with the \textit{letrec-height} at which they are applied. The \textit{letrec}-height of a position in \( \lambda \textit{letrec} \)-term denotes the maximal number of \textit{let}-nodes passed on the path from the root of the term tree to any of its leaves.
First, we observe that $R_\triangledown$’s rules never increase the $\text{letrec}$-height of the term which renders the indexing admissible in the sense that it leaves the number of rules finite. Furthermore we find that for any $\lambda\text{letrec}$-term $\rho_{\triangledown}\text{rec}$ and $\rho_{\triangledown}\text{Nil}$ cannot be applied at the same level and the rule applied at a higher level (i.e. at lower $\text{letrec}$-height) leads to a duplication of the other rule in the diagram. Therefore we can order the rules as follows. Let $\rho_{\triangledown}^a$ and $\rho_{\triangledown}^b$ be indexed rules out of the set $\{\rho_{\triangledown}\text{rec}, \rho_{\triangledown}\text{Nil}\}$ and let $\rho_{\triangledown}^c$ be a rule not in that set. Then we demand of the order:

- $\rho_{\triangledown}^a > \rho_{\triangledown}^c$ and $\rho_{\triangledown}^b > \rho_{\triangledown}^c$
- $\rho_{\triangledown}^a > \rho_{\triangledown}^b$ if the index of $\rho_{\triangledown}^a$ is greater than the index of $\rho_{\triangledown}^b$

Example 3.12. $R_\triangledown$ when applied to $\text{letrec } f = \lambda xy. fy x$ in $f$ admits the following rewrite sequence:

$$
\begin{align*}
\text{letrec } f &= \lambda xy. fy x \\
\to_{\triangledown, \text{rec}} \text{letrec } f &= \lambda xy. fy x \in \lambda xy. fy x \\
\to_{\triangledown, \lambda} \lambda x. \text{letrec } f &= \lambda xy. fy x \in \lambda y. fy x \\
\to_{\triangledown, \lambda} \lambda xy. \text{letrec } f &= \lambda xy. fy x \in \lambda y. fy x \\
\to_{\triangledown, \alpha} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f y) (\text{letrec } f = \lambda xy. fy x \in x) \\
\to_{\triangledown, \text{red}} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f y) (\text{letrec } \in x) \\
\to_{\triangledown, \text{nil}} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f y) x \\
\to_{\triangledown, \alpha} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f) (\text{letrec } f = \lambda xy. fy x \in y) x \\
\to_{\triangledown, \text{red}} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f) (\text{letrec } \in y) x \\
\to_{\triangledown, \text{nil}} \lambda xy. (\text{letrec } f = \lambda xy. fy x \in f) y x \\
\to_{\triangledown, \text{rec}} \lambda xy. \ldots y x
\end{align*}
$$

However, not every $\lambda\text{letrec}$-terms unfolds to an infinite $\lambda$-term in the sense that it has an infinite $\lambda$-term as its infinite $\to_{\triangledown}$-normal form. For example, the $\lambda\text{letrec}$-term $L = \text{letrec } f = f$ in $f$ admits only rewrite sequences of the form $L \to_{\triangledown, \text{rec}} L \to_{\triangledown, \text{rec}} \ldots$, and hence does not unfold to an infinite $\lambda$-term. Terms like this are unproductive in the sense that during all outermost-fair rewrite sequences the production of an infinite $\lambda$-term stagnates due to an unproductive cycle.

Definition 3.13 ($R_\triangledown$-productivity). Let $L$ be a $\lambda\text{letrec}$-term $L$. We say that $L$ is $R_\triangledown$-productive if the following equivalent statements hold:

- $L$ does not have a $\to_{\triangledown}$-reduct that is the source of an infinite $\to_{\triangledown}$-rewrite sequence consisting exclusively of outermost steps with respect to $\to_{\triangledown, \text{rec}}, \to_{\triangledown, \text{nil}}, \to_{\triangledown, \text{letrec}}$, or $\to_{\triangledown, \text{red}}$.
- if every $\to_{\triangledown}$-rewrite sequence starting with a $\to_{\triangledown}$-reduct contains infinitely many outermost $\to_{\triangledown, \lambda}$- or $\to_{\triangledown, \alpha}$-steps.
Lemma 3.14. Let \( L \) be a \( \lambda_{letrec} \)-term of the form \( \text{let}_n([\bar{f}] \text{rec-in}_n(P_1, \ldots, P_{n-1}, P_n)) \). The following statement holds either for all maximal outermost-fair \( R_\varphi \) rewriting sequences \( \tau \) on \( L \) or for none: \( \tau \) contains an outermost \( \rightarrow_\varphi,\lambda \)- or \( \rightarrow_\varphi,\@ \)-step.

As a consequence either all but finitely many terms of the sequence are \( \lambda \)-abstractions or applications, respectively, or all terms of the sequence start with an outermost \( \text{letrec} \).

Lemma 3.15. For all \( \lambda_{letrec} \)-terms \( L \) the following statements are equivalent:

(i) \( L \rightarrow^\omega_\varphi M \) for some infinite \( \lambda \)-term \( M \).

(ii) \( L \) is \( R_\varphi \)-productive.

(iii) Every maximal outermost-fair \( \rightarrow_\varphi \)-rewrite sequence on \( L \) is strongly convergent.

Proof. (ii) \( \Rightarrow \) (iii), because if \( L \) is \( R_\varphi \)-productive then every outermost occurrence of a \( \text{letrec} \) in every \( \rightarrow_\varphi \)-reduct will be eventually pushed down to a higher position by either a \( \rightarrow_\varphi,\lambda \)- or a \( \rightarrow_\varphi,\@ \)-step of any maximal outermost-fair \( \rightarrow_\varphi \)-sequence. Since only \( \text{letrec} \)-terms are redexes in \( R_\varphi \) any maximal outermost-fair rewrite sequence starting from \( L \) converges to an infinite normal form. (i) follows directly from (iii). (i) \( \Rightarrow \) (ii) follows from Lemma 3.14 by contradiction. If \( L \) is not \( R_\varphi \)-productive then it has by definition a \( \rightarrow_\varphi \)-reduct with at least one occurrence of a \( \text{letrec} \) which cannot be pushed further down by any outermost application of any \( R_\varphi \)-rule. By Lemma 3.14 the same holds for every other maximal outermost-fair \( \rightarrow_\varphi \)-rewrite sequence. Therefore \( L \) cannot unfold to an infinite \( \lambda \)-term \( M \) because \( M \) may not contain any \( \text{letrec} \)s.

Lemma 3.16 (uniqueness of unfolding). Unfolding normal forms of \( \lambda_{letrec} \)-terms reachable in at most \( \omega \) steps are unique. That is: if \( M_1 \leftarrow_\varphi^\omega L \rightarrow_\varphi^\omega M_2 \) for a \( \lambda_{letrec} \)-term \( L \), and infinite \( \lambda \)-terms \( M_1 \) and \( M_2 \), then \( M_1 = M_2 \).

Proof sketch. Let us assume that \( L \) unfolds to \( M_1 \) and \( M_2 \) by the \( \rightarrow_\varphi \)-reduction sequences \( \tau_1 \) and \( \tau_2 \). Consider for \( n \in \mathbb{N} \) the first reduct \( P_1 \) (or \( P_2 \)) in the sequence \( \tau_1 \) (or \( \tau_2 \)) that is stable above depth \( n \). It follows from confluence of the rewrite relation (Proposition 3.11) that \( P_1 \) and \( P_2 \) have a common \( \rightarrow_\varphi \)-reduct \( P_3 \). Since the rules of \( R_\varphi \) do not create a redex at lower depth than the occurrence of the left-hand side (‘redexes are not pushed upwards’) in the rewriting sequence from \( P_1 \) (or \( P_2 \)) to \( P_3 \), no contractions take place above depth \( n \). In that sense \( P_1 \) and \( P_2 \) are ‘equal up to depth \( n \)’. Such a notion is, however, still in need of precise formulation for CRS-terms. Argument can be repeated on arbitrary \( n \), \( M_1 \) and \( M_2 \) agree on arbitrarily large outermost contexts. \( M_1 = M_2 \). Still, for the purpose of this proof sketch we conclude by this first-order argument.

As a consequence of the lemma above, the rewriting system \( R_\varphi \) defines a partial function \( U \) for unfolding \( \lambda_{letrec} \)-terms.

Definition 3.17 (unfolding as a mapping). We define the partial unfolding function:

\[ U : \text{Ter}(\lambda_{letrec}) \rightarrow \text{Ter}(\lambda^\omega) \]

\[ L \mapsto M \quad \text{if} \quad L \rightarrow_\varphi^\omega M \]

We say that \( L \) expresses \( M \) if \( U(L) = M \). Uniqueness of \( U \) follows from Lemma 3.15.
Example 3.18. The terms from Example 1.2 both express the same $\lambda^\infty$-term:
\[ \lambda x. \text{letrec } r \text{ in } x : x : x : \ldots \rightarrow_{\varphi} \lambda x. \text{letrec } r \text{ in } (x : r) \]

Example 3.19 ($R_\varphi$-unproductive $\lambda_{\text{letrec}}$-term). As an example for a non-unfoldable $\lambda_{\text{letrec}}$-term, consider $\text{letrec } f = \text{letrec } g = f \text{ in } g \text{ in } f$ which is not in $\text{dom}(U)$ and the cyclic rewriting sequence:

\[
\begin{align*}
\text{letrec } f &= \text{letrec } g = f \text{ in } g \text{ in } f \\
\rightarrow_{\varphi, \text{rec}} &\quad \text{letrec } f = \text{letrec } g = f \text{ in } \text{letrec } g = f \text{ in } g \\
\rightarrow_{\varphi, \text{letrec}} &\quad \text{letrec } f = \text{letrec } g = f \text{ in } g ; g = f \text{ in } f \\
\rightarrow_{\varphi, \text{rec}} &\quad \text{letrec } f = \text{letrec } g = f \text{ in } g
\end{align*}
\]

We will revisit this example in Example 6.21 to illustrate the cyclicity proof.

We also define an unfolding function which is complete on $\text{Ter}(\lambda_{\text{letrec}})$ by mapping non-unfoldable subterms to $\bot$, which yields Böhm trees in $\lambda^\infty$.

Definition 3.20 (partial unfolding). $\text{Ter}_1(\lambda^\infty)$ denotes infinite terms over $\Sigma_\lambda \cup \{1\}$, where 1 is a constant symbol.

\[ U' : \text{Ter}(\lambda_{\text{letrec}}) \rightarrow \text{Ter}_1(\lambda^\infty) \]

\[ L \rightarrow M \text{ if } L \rightarrow_{\varphi} M \]

Thereby $\rightarrow_{\varphi}$ is the infinitary rewrite relation induced by the rewrite relation $\rightarrow_1$ that extends $\rightarrow_{\varphi}$ by mapping $R_\varphi$-root-active subterms to 1.

4. Regular and strongly regular terms in $\lambda^\infty$

For infinite first-order trees the concept of regularity is well-known and well-studied (Courcelle, 1983). Regularity of a labelled tree is defined as the existence of only finitely many subtrees and implies the existence of a finite graph that unfolds to that tree. In this section we generalise the notion of regularity to trees with a binding mechanism, the $\lambda^\infty$-calculus specifically. We give a definition for regularity which corresponds to regularity of a term when regarded as a first-order tree, and for strong regularity, which will be shown in the following sections to coincide with $\lambda_{\text{letrec}}$-expressibility.

We define regularity and strong regularity in terms of rewriting systems that will be called $\text{Reg}$ and $\text{Reg}^*$. Rewrite sequences in these systems inspect a given term coinductively in the sense that a rewrite sequence corresponds to a decomposition of the term along one of its paths from the root. Both $\text{Reg}$ and $\text{Reg}^*$ both extend a kernel system $\text{Reg}^-$ comprising three rewrite rules which denote whether the position just passed in the tree is an abstraction or an application and in the second case whether the application is being followed to the left or to the right.

\[ \text{Reg}^- \]

By a 'labelled tree' we here mean a finite or infinite tree whose nodes are labelled by function symbols from a first-order signature such that the arity of the function symbol in a node determines the number of successors of the node.
The rewriting systems are defined on $\lambda$-terms enriched by what we call an *abstraction prefix*, by which the terms can be kept closed during the deconstruction. This is crucial for the definition of the rewriting system as a CRS. While intuitively it is clear that the $\lambda$-term $MN$ is composed of the subterms $M$ and $N$, abstractions are more problematic. In a first-order setting one could say that $\lambda x.M$ contains $M$ as a subterm, but if $x$ occurs freely in $M$ then $M$ would be an open term. That means that the scrutinisation of an abstraction would be able go from a closed term to an open term, which would run counter to the interpretation of a higher-order term as an $\alpha$-equivalence class. In the definition of the rewriting systems below this issue is resolved as follows. When inspecting an abstraction the binder is not left out but moved from the scrutinised subterm into the prefix. That guarantees that the term as a whole remains closed.

In $\lambda x.\lambda y.xxy$ for instance the path from the root to the second occurrence of $x$ then corresponds to the rewrite sequence:

$$(\lambda x.\lambda y.xxy) \rightarrow (\lambda x)(\lambda xy)x x y \rightarrow (\lambda xy)x x \rightarrow x \rightarrow S(\lambda x)x$$

The $\text{Reg}^+$ and the $\text{Reg}$ system extend $\text{Reg}^-$ by a *scope-delimiting* rule, which signifies that the scope of an abstraction has ended, whereby both systems are based on different notions of scope.

$\text{Reg}$ relies on what we simply call *scope*, which the scope of an abstraction denotes the range from the abstraction up to the points under which the bound variable does not occur anymore.

We base $\text{Reg}^+$ on a different notion of scope, called *scope* $^+$, which is strictly nested. The scope $^+$s of an abstraction ranges extends its scope by encompassing all scope $^+$s that are opened within its range. As a consequence scope $^+$s do no overlap (see Figure 5).

Precise definitions of scope and scope $^+$ are given later in Definition 7.2.

When every scope $^+$ is closed by the scope-delimiting rule then the sequence of rewrite steps alone (i.e. without the terms themselves) unambiguously determines which abstraction a variable occurrence belongs to. The rewrite sequence from above would then have one additional scope $^+$-delimiting step asserting that the variable at the end of the path is indeed $x$ and not $y$:

$$(\lambda x.\lambda y.xxy) \rightarrow (\lambda x)(\lambda xy)x x y \rightarrow (\lambda xy)x x \rightarrow (\lambda xy)x \rightarrow x \rightarrow S(\lambda x)x$$

The abstraction prefix not only keeps the term closed but also denotes which scope $^+$ is still open, which provides the information to decide applicability of the scope-delimiting
rule. The last step closes the scope* of y, therefore that variable is removed from the prefix. The rewrite sequence for the path to the occurrence of y does not include an scope*-delimiting step:

\((\lambda x.\lambda y. x \shuffle y) \rightarrow_{\lambda} (\lambda x) \lambda y. x \shuffle y \rightarrow_{\lambda} (\lambda x) y\)

Ultimately, the \(\text{Reg}^+\) rewriting system defines nameless representations for \(\lambda\)-terms related to the de-Bruijn notation. Considering the de-Bruin representation of the above term \(\lambda x(S) 0\) we find that the position of the \(\rightarrow_{\text{S}}\)-steps indeed coincides with the position of the S markers. The \(\text{Reg}^+\) does however permits more flexibility with the placement of \(\rightarrow_{\text{S}}\)-steps, cp. (van Oostrom et al., 2004). For example the path from above to the second occurrence of x can also be witnessed by another rewrite sequence

\((\lambda x.\lambda y. x \shuffle y) \rightarrow_{\lambda} (\lambda x) \lambda y. x \shuffle y \rightarrow_{\lambda} (\lambda x) x \shuffle x \rightarrow_{\lambda} (\lambda x) x\)

where the scope* of y is closed earlier. This would correspond to \(\lambda x(S(0)) 0\) in de-Bruijn notation, that is to say in a variant of the de-Bruijn notation which allows for S to be used distributively.

**Definition 4.1 (CRS-terms with abstraction prefixes).** The CRS-signature for \(\lambda\), the \(\lambda\)-calculus with abstraction prefixes, extends the CRS-signature \(\Sigma_\lambda\) for \(\lambda\) (see Definition 3.3) and consists of the set \(\Sigma_{\lambda(\lambda)}\) = \(\Sigma_\lambda \cup \{\text{pre}_n | n \in \mathbb{N}\}\) of function symbols, where for \(n \in \mathbb{N}\) the function symbols \(\text{pre}_n\) for prefix \(\lambda\)-abstractions of length \(n\) are unary (have arity one). Using the syntactical variables for terms in \(\lambda\), CRS-terms with leading prefixes \(\text{pre}_n([x_1] \ldots [x_n])M\) will informally be denoted by \((\lambda x_1 \ldots x_n) M\), abbreviated as \((\lambda x) M\).

**Definition 4.2 (The CRSs \text{Reg}^-, \text{Reg}, \text{Reg}^+ for decomposing \(\lambda\)-terms).** Consider the following CRS-rules over signature \(\Sigma_{\lambda(\lambda)}\):

\[
\begin{align*}
(\varrho^0) : & \quad \text{pre}_n([x_1] \ldots [x_n]) \text{app}(Z_0(\bar{x}), Z_1(\bar{x})) \rightarrow \text{pre}_n([x_1] \ldots [x_n]) Z(\bar{x}) \\
(\varrho^\lambda) : & \quad \text{pre}_n([x_1] \ldots [x_n]) \text{abs}([x_{n+1}] Z(\bar{x})) \rightarrow \text{pre}_n([x_1] \ldots [x_{n+1}]) Z(\bar{x}) \\
(\varrho^5) : & \quad \text{pre}_{n+1}([x_1] \ldots [x_n]) Z(x_1, \ldots, x_n) \rightarrow \text{pre}_{n}([x_1] \ldots [x_n]) Z(x_1, \ldots, x_n) \\
(\varrho^\text{del}) : & \quad \text{pre}_{n+1}([x_1] \ldots [x_{n+1}]) Z(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \rightarrow \text{pre}_{n}([x_1] \ldots [x_{i-1} x_{i+1}, \ldots, x_n]) Z(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\end{align*}
\]

By \(\text{Reg}^-\) we denote the CRS with rules \(\varrho^0\) and \(\varrho^\lambda\). By \(\text{Reg}\) (and respectively, by \(\text{Reg}^+\)) we denote the CRS consisting of all of the above rules except the rule \(\varrho^5\) (except the rule \(\varrho^\text{del}\)). The rewrite relations of \(\text{Reg}^-, \text{Reg}, \text{Reg}^+\) are denoted by \(\rightarrow_{\text{reg}^-}, \rightarrow_{\text{reg}}\) and \(\rightarrow_{\text{reg}^+}\), respectively. And by \(\rightarrow_{\varrho^0}, \rightarrow_{\varrho^\lambda}, \rightarrow_{\varrho^5}, \rightarrow_{\varrho^\text{del}}\) we respectively denote the rewrite relations induced by each of the single rules \(\varrho^0, \varrho^\lambda, \varrho^5, \varrho^\text{del}\).

Assuming that the translation between the formal and the informal notation is facile, for better unreadability we will from now on rely on the latter. Here are the rules from above in informal notation:

\[
\begin{align*}
(\varrho^0) : & \quad (\lambda x_1 \ldots x_n) M_0 M_1 \rightarrow (\lambda x_1 \ldots x_n) M_i \quad (i \in \{0, 1\}) \\
(\varrho^\lambda) : & \quad (\lambda x_1 \ldots x_n) \lambda x_{n+1}. M_0 \rightarrow (\lambda x_1 \ldots x_{n+1}) M_0
\end{align*}
\]
Fig. 6. The sub-ARSs induced by () \( \lambda x. \lambda y. x \cdot y \). Note, that in the pictures the nodes do not display the entire \( (\lambda^\omega) \)-terms but only their prefixes.

\[
(\gamma^S) : \quad (\lambda x_1 \ldots x_{n+1}) M_0 \rightarrow (\lambda x_1 \ldots x_n) M_0 \quad \text{(if the binding } \lambda x_{n+1} \text{ is vacuous)}
\]

\[
(\gamma^{del}) : \quad (\lambda x_1 \ldots x_{n+1}) M_0 \rightarrow (\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1}) M_0
\]

\text{(if the binding } \lambda x_i \text{ is vacuous)}

**Remark 4.3 (only } \text{Reg}^+ \text{ defines nameless representations).** Considering the graphs from Figure 6 without labels on their nodes we see that only only from the } \text{Reg}^+ \text{ graph the original term could be reconstructed unambiguously. The path to the first occurrence of } x \text{ for instance has the rewrite sequence in } \text{Reg}

\[
\rightarrow_{\lambda} \cdot \rightarrow_{\lambda} \cdot \rightarrow_{@0} \cdot \rightarrow_{S} \cdot \rightarrow_{@1}
\]

which would also be an admissible to witness an occurrence of } y \text{ at the same position. This ambiguity is discussed later in Remark 9.9.

Note that the following relationships between rewrite relations: } \rightarrow_{\text{reg}} \text{ is contained in } \rightarrow_{\text{reg}\,^-} \text{ and } \rightarrow_{\text{reg}\,^+}, \text{ and since the rule } \gamma^{\text{del}} \text{ generalises the rule } \gamma^S, \rightarrow_{\text{reg}\,^+} \text{ is contained in } \rightarrow_{\text{reg}}.

**Proposition 4.4.** Every rewrite sequence in } \text{Reg}^+ \text{ corresponds directly to a rewrite sequence in } \text{Reg} \text{ by exchanging } \rightarrow_{S} \text{-steps with } \rightarrow_{\text{del}} \text{-steps.}

Our interest will focus on the subset of terms with an outermost abstraction prefix symbol and no other occurrences of such symbols. Note that the rules in } \text{Reg} \text{ and } \text{Reg}^+ \text{ guarantee that every reduct of a term of the form } (\lambda \vec{x}) M \text{ is again a term of this form. Therefore we define:}

**Definition 4.5 (prefixed } \lambda^\omega \text{-terms).** By } \text{Ter}( (\lambda^\omega) ) \text{ we denote the set of closed iCRS-terms over } \Sigma_{(\lambda)} \text{ with the restriction that}
— every term \( M \in \text{Ter}(\mathcal{L}_\infty) \) has a prefix at its root and nowhere else, or in other words: \( M \) is of the form \( \text{pre}_n([x_1]...[x_n]M) \) where \( M \) does not contain any occurrences of function symbols \( \text{pre}_i \) for \( i \in \mathbb{N} \nabla \)

— and that otherwise a CRS abstraction can only occur directly beneath an \( \text{abs} \) symbol.

\( \text{Ter}(\mathcal{L}_\infty) \) is more formally specified in Definition 6.2.

**Proposition 4.6.** \( \text{Ter}(\mathcal{L}_\infty) \) is closed under \( \rightarrow_{\text{reg}}, \leftarrow_{\text{reg}}, \) and \( \rightarrow_{\text{reg}^*} \).

**Definition 4.7 (the ARSs \( \text{Reg}^-, \text{Reg}, \text{Reg}^* \)).** We denote by \( \text{Reg}^-, \text{Reg} \) and \( \text{Reg}^* \) the infinite abstract rewriting systems (ARSs) induced by the iCRSs derived from \( \text{Reg}^-, \text{Reg}, \text{Reg}^* \), restricted to terms in \( \text{Ter}(\mathcal{L}_\infty) \).

The rewrite relations of \( \text{Reg}^-, \text{Reg} \) and \( \text{Reg}^* \) will be denoted by the same symbols used for \( \text{Reg}^-, \text{Reg} \) and \( \text{Reg}^* \). Since all of our considerations will refer to the restricted set of terms, this should create no confusion.

**Proposition 4.8.** The restrictions of the rewrite relations as defined in Definition 4.2 to \( \text{Ter}(\mathcal{L}_\infty) \), the set of objects of \( \text{Reg}^-, \text{Reg}, \) and \( \text{Reg}^* \), have the following properties:§

(i) \( \rightarrow_{\text{del}} \) is confluent, and terminating. \((\lambda x)x\) is the only term in \( \rightarrow_{\text{del}} \)-normal form.

(ii) \( \rightarrow_{\text{del}} \) one-step commutes with \( \rightarrow_{\lambda}, \rightarrow_{@_0}, \rightarrow_{@_1} \), and one-step sub-commutes with \( \rightarrow_{S} \):

\[
\begin{align*}
\rightarrow_{\text{del}} \cdot \rightarrow_{\lambda} & \subseteq \rightarrow_{\lambda} \cdot \rightarrow_{\text{del}} \quad \left( i \in \{0, 1\} \right) \\
\rightarrow_{\text{del}} \cdot \rightarrow_{S} & \subseteq \rightarrow_{S} \cdot \rightarrow_{\text{del}} \\
\end{align*}
\]

(iii) \( \rightarrow_{\text{del}} \subseteq \rightarrow_{\text{del}} \), and consequently, \( \rightarrow_{\text{reg}} \subseteq \rightarrow_{\text{reg}} \). Furthermore, \( \rightarrow_{\text{reg}} \subseteq \rightarrow_{\text{reg}^*} \subseteq \rightarrow_{\text{reg}} \).\n
(iv) \( \rightarrow_{S} \) is deterministic, hence confluent, and terminating. Every term in \( \rightarrow_{S} \)-normal form is of the form \((\lambda x_1...x_n)x_n\), and \( \rightarrow_{\text{reg}} \):

\[
\begin{align*}
\rightarrow_{S} \cdot \rightarrow_{\lambda} & \subseteq \rightarrow_{\lambda} \cdot \rightarrow_{S} \quad \left( \lambda \in \{0, 1\} \right) \\
\rightarrow_{S} \cdot \rightarrow_{@_1} & \subseteq \rightarrow_{@_1} \cdot \rightarrow_{S} \\
\end{align*}
\]

(v) \( \rightarrow_{S} \) one-step commutes with \( \rightarrow_{\lambda}, \rightarrow_{@_0}, \) and \( \rightarrow_{@_1} \):

\[
\begin{align*}
\rightarrow_{S} \cdot \rightarrow_{\lambda} & \subseteq \rightarrow_{\lambda} \cdot \rightarrow_{S} \\
\rightarrow_{S} \cdot \rightarrow_{@_1} & \subseteq \rightarrow_{@_1} \cdot \rightarrow_{S} \\
\end{align*}
\]

\( (\lambda x)x \) is the sole term in \( \rightarrow_{\text{reg}} \)-normal form. Every \( \rightarrow_{\text{reg}^*} \)-normal form is of the form \((\lambda x_1...x_n)x_n\).

(vi) \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^*} \) are finitely branching, and, on finite terms, terminating.

**Proof.** Most properties, including those concerning commutation of steps, are easy to verify by analysing the behaviour of the rewrite rules in \( \text{Reg} \) on terms of \( \text{Ter}(\mathcal{L}_\infty) \).

For \( \rightarrow_{\text{reg}} \subseteq \rightarrow_{\text{reg}^*} \) in (iii) note that, for example, \((\lambda xy)y \rightarrow_{\text{reg}} (\lambda y)y \) by a \( \rightarrow_{\text{del}} \)-step, but that \((\lambda xy)y \) is a \( \rightarrow_{S} \)-normal form, and hence also a \( \rightarrow_{\text{reg}^*} \)-normal form.

In item (vi) we first argue for finite branchingness of \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^*} \) on \( \text{Ter}(\mathcal{L}_\infty) \): this property is entailed by the fact that, on a term \((\lambda x)M \) with just one abstraction in its prefix, of the constituent rewrite relations \( \rightarrow_{@_0}, \rightarrow_{@_1}, \rightarrow_{\lambda}, \rightarrow_{S}, \rightarrow_{\text{del}} \) of \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^*} \), only \( \rightarrow_{\text{del}} \) can have branching degree greater than one, which in this case then also is bounded by the length \(|\vec{z}|\) of the abstraction prefix. For termination of \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^*} \) on finite terms with just a leading abstraction prefix we can restrict to \( \rightarrow_{\text{reg}} \), due to (iii), and argue as follows: On finite terms in \( \text{Ter}(\mathcal{L}_\infty) \), in every \( \rightarrow_{\text{reg}} \)-rewrite step either the

§ Mind the restriction here to terms in \( \text{Ter}(\mathcal{L}_\infty) \).
size of the body of the term decreases strictly, or the size of the body stays the same, but the length of the prefix decreases by one. Hence in every rewrite step the measure (body size, prefix length) on terms decreases strictly in the (well-founded) lexicographic ordering on \(\mathbb{N} \times \mathbb{N}\).

As a consequence of Proposition 4.8, (i) and (iv), the rewrite relations \(\rightarrow_{\text{del}}\) and \(\rightarrow_{5}\) are normalizing on \(\text{Ter}(\lambda^\infty)\). For every term \(M \in \text{Ter}(\lambda^\infty)\) we will denote by \(M_{\text{del}}\) and \(M_{\text{del}}\) the normal forms of \(M\) with respect to \(\rightarrow_{\text{del}}\) and \(\rightarrow_{5}\), respectively. And by \(\rightarrow_{1}\) (by \(\rightarrow_{1}\)) we denote the many-step rewrite relation for \(\rightarrow_{\text{del}}\) (for \(\rightarrow_{5}\)) that leads to a \(\rightarrow_{\text{del}}\)-normal form (to a \(\rightarrow_{5}\)-normal form), that is: \(M \rightarrow_{\text{del}} N\) if \(M_{\text{del}} = N\) (and respectively, \(M \rightarrow_{5} N\) if \(M_{\text{del}} = N\), for all terms \(M\) and \(N\).

**Proposition 4.9.** The following statements hold:

(i) Let \((\lambda x)M\) a term in \(\text{Reg}\) with \(|\tilde{x}| = n \in \mathbb{N}\). Then the number of terms \((\lambda \tilde{y})N\) in \(\text{Reg}\) with \((\lambda \tilde{y})N \rightarrow_{\text{del}} (\lambda x)M\) and \(|\tilde{y}| = n + k \in \mathbb{N}\) is \(\binom{n+k}{k}\).

(ii) Let \(T\) be a finite set of terms in \(\text{Reg}\), and \(l \in \mathbb{N}\). Then also the set of terms in \(\text{Reg}\) that are the form \((\lambda \tilde{y})N\) with \(|\tilde{y}| \leq k\) and that have a \(\rightarrow_{\text{del}}\)-reduct in \(T\) is finite.

**Proof.** If \((\lambda \tilde{y})M(y) = (\lambda y_1 \ldots y_{n+k})N(y_1, \ldots, y_{n+k}) \rightarrow_{\text{del}} (\lambda x_1 \ldots x_n)M(x_1, \ldots, x_n) = (\lambda \tilde{x})M(\tilde{x})\), then it follows that there are \(i_1, \ldots, i_n \in \{1, \ldots, n + k\}\) with \(i_1 < i_2 < \ldots < i_n\) such that the term \((\lambda \tilde{y})M(y)\) is actually of the form \((\lambda y_1 \ldots y_{n+k})N(y_1, \ldots, y_{n+k})\) and furthermore \((\lambda y_1 \ldots y_{n})N(y_1, \ldots, y_{n}) = (\lambda x_1 \ldots x_n)M(x_1, \ldots, x_n)\). Hence the number of terms \((\lambda \tilde{y})M(y)\) with \(\rightarrow_{\text{del}}\)-reduct \((\lambda \tilde{x})M(\tilde{x})\) is equal to the number of choices \(i_1, \ldots, i_n \in \{1, \ldots, n + k\}\) such that \(i_1 < i_2 < \ldots < i_n\). This establishes statement (i). Statement (ii) is an easy consequence.

**Lemma 4.10.** On \(\text{Ter}(\lambda^\infty)\), the rewrite relations \(\rightarrow_{\text{reg}}\) and \(\rightarrow_{\text{reg}}\) have the following further properties with respect to \(\rightarrow_{\text{del}}\), \(\rightarrow_{5}\), \(\rightarrow_{\text{del}}\), and \(\rightarrow_{5}\):

\[
\begin{align*}
&\leftarrow_{\text{del}} \cdot \rightarrow_{\text{reg}} \subseteq \left(\rightarrow_{\text{del}} \cdot \rightarrow_{5}^{-}\right) \cdot \leftarrow_{\text{del}} \quad (4.1) \\
&\leftarrow_{\text{del}} \cdot \rightarrow_{\text{reg}} \supseteq \left(\rightarrow_{\text{del}} \cdot \rightarrow_{\text{reg}}\right)^{-} \cdot \leftarrow_{\text{del}} \quad (4.2) \\
&\leftarrow_{\text{del}} \cdot \rightarrow_{\text{reg}}^{-} \subseteq \left(\rightarrow_{\text{del}} \cdot \rightarrow_{\text{reg}}^{-}\right) \cdot \leftarrow_{\text{del}} \quad (4.3) \\
&\leftarrow_{\text{del}} \cdot \rightarrow_{\text{reg}}^{-} \supseteq \left(\rightarrow_{\text{del}} \cdot \rightarrow_{\text{reg}}^{-}\right)^{-} \cdot \leftarrow_{\text{del}} \quad (4.4)
\end{align*}
\]

**Proof.** These commutation properties, which can be viewed as projection properties, can be shown by arguments with diagrams using the commutation properties in Proposition 4.8.

**Remark 4.11.** The commutation properties in Proposition 4.10 can be refined to state that \(\rightarrow_{\lambda}\)-steps project to \(\rightarrow_{\lambda}\)-steps, and \(\rightarrow_{\@}\)- and \(\rightarrow_{\@}\)-steps project to \(\rightarrow_{\@}\)- and \(\rightarrow_{\@}\)-steps, accordingly.

As an immediate consequence of Proposition 4.10 we obtain the following lemma, which formulates a connection via projection between rewrite sequences in \(\text{Reg}\) (in \(\text{Reg}'\)) and \(\rightarrow_{\text{del}}\)-eager (\(\rightarrow_{5}\)-eager) rewrite sequences in \(\text{Reg}\) (in \(\text{Reg}'\)) that do not contain \(\rightarrow_{\lambda}, \rightarrow_{\@}\), or \(\rightarrow_{\@}\)-steps on terms that allow \(\rightarrow_{\text{del}}\)-steps (\(\rightarrow_{5}\)-steps).
Lemma 4.12. The following statements hold:

(i) Every (finite or infinite) rewrite sequence in $\text{Reg}$ of the form:
\[
\tau : (\lambda \tilde{x}_0) M_0 \rightarrow_{\text{reg}} (\lambda \tilde{x}_1) M_1 \rightarrow_{\text{reg}} \ldots \rightarrow_{\text{reg}} (\lambda \tilde{x}_k) M_k \rightarrow_{\text{reg}} \ldots
\]
projects over a rewrite sequence $\pi : (\lambda \tilde{x}_0) M_0 \rightarrow_{\text{del}} (\lambda \tilde{x}_0') M_0$ to a $\rightarrow_{\text{del}}$-eager rewrite sequence in $\text{Reg}$ of the form:
\[
\hat{\tau} : (\lambda \tilde{x}_0') M_0 \rightarrow_{\text{del}}^1 \rightarrow_{\text{reg}}^\ast (\lambda \tilde{x}_1') M_1 \rightarrow_{\text{del}}^1 \rightarrow_{\text{reg}}^\ast \ldots
\]
\[
\ldots \rightarrow_{\text{del}}^1 \rightarrow_{\text{reg}}^\ast (\lambda \tilde{x}_k') M_k \rightarrow_{\text{del}}^1 \rightarrow_{\text{reg}}^\ast \ldots
\]
in the sense that $(\lambda \tilde{x}_i) M_i \rightarrow_{\text{del}} (\lambda \tilde{x}_i') M_i$ for all $i \in \mathbb{N}$ less or equal to the length of $\tau$.

(ii) Every (finite or infinite) rewrite sequence in $\text{Reg}^\ast$ of the form:
\[
\tau : (\lambda \tilde{y}_0) N_0 \rightarrow_{\text{reg}}^\ast (\lambda \tilde{y}_1) N_1 \rightarrow_{\text{reg}}^\ast \ldots \rightarrow_{\text{reg}}^\ast (\lambda \tilde{y}_k) N_k \rightarrow_{\text{reg}}^\ast \ldots
\]
projects over a rewrite sequence $\pi : (\lambda \tilde{y}_0) N_0 \rightarrow_{\text{del}} (\lambda \tilde{y}_0') N_0$ to a $\rightarrow_{\text{s}}$-eager rewrite sequence in $\text{Reg}^\ast$:
\[
\hat{\tau} : (\lambda \tilde{y}_0') N_0 \rightarrow_{\text{s}}^1 \rightarrow_{\text{reg}}^\ast (\lambda \tilde{y}_1') N_1 \rightarrow_{\text{s}}^1 \rightarrow_{\text{reg}}^\ast \ldots
\]
\[
\ldots \rightarrow_{\text{s}}^1 \rightarrow_{\text{reg}}^\ast (\lambda \tilde{y}_k') N_k \rightarrow_{\text{s}}^1 \rightarrow_{\text{reg}}^\ast \ldots
\]
in the sense that $(\lambda \tilde{y}_i) N_i \rightarrow_{\text{s}} (\lambda \tilde{y}_i') N_i$ for all $i \in \mathbb{N}$ less or equal to the length of $\tau$.

Remark 4.13 (non-determinism of $\rightarrow_{\text{reg}}$ and $\rightarrow_{\text{reg}}^\ast$). On terms in $\text{Ter}(\text{Reg}^\ast)$, which have just one prefix at the top of the term, there are two different causes for non-determinism of the rewrite relations in $\text{Reg}$ and $\text{Reg}^\ast$: First, since the left-hand sides of the rules $\varrho^{\text{det}}$ and $\varrho^{\text{del}}$, coincide, these rules enable different steps on the same term, producing the left- and respectively the right subterm of the application immediately below the prefix. Second, the rules $\varrho^{\text{del}}$ and $\varrho^{\text{s}}$ can be applicable in situations where also one of the rules $\varrho^{\text{det}}$, $\varrho^{\text{del}}$, or $\varrho^{\text{s}}$ is applicable. Whereas the first kind of non-determinism is due to the ‘observer’ having to observe the two different subterms of an application in a $\lambda$-term, the second is due to a freedom of the observer as to when to attest the end of a scope (of some kind) in the analysed $\lambda$-term.

In the definition below we define strategies on $\text{Reg}$ and $\text{Reg}^\ast$ that resolves the second source of non-determinism while leaving the first kind intact. As a result the sub-ARS induced by some term $M$ with respect to $\text{Reg}^\ast$ correspond structurally to the term graph of $M$.

Definition 4.14 (scope/scope$^\ast$-delimiting strategy). We call a strategy $\mathcal{S}$ for $\text{Reg}$ (for $\text{Reg}^\ast$) a scope-delimiting strategy (a scope$^\ast$-delimiting strategy) if the source of a step is non-deterministic (that is, it is the source of more than one step) if and only if it is the source of precisely a $\rightarrow_{\text{del}}$-step and a $\rightarrow_{\text{det}}$-step.

For every such strategy $\mathcal{S}$, we denote by $\rightarrow_{\text{det}}\mathcal{S}$, $\rightarrow_{\text{det}}\mathcal{S}$, $\rightarrow_{\text{det}}\mathcal{S}$, $\rightarrow_{\text{det}}\mathcal{S}$, and $\rightarrow_{\text{det}}\mathcal{S}$, the rewrite relations that are induced by those steps according to $\mathcal{S}$ that are induced by applications of the rules $\varrho^{\text{det}}$, $\varrho^{\text{del}}$, $\varrho^{\text{s}}$, and $\varrho^{\text{det}} \rightarrow_{\text{det}}\mathcal{S}$ (for $\rightarrow_{\text{det}}\mathcal{S}$), respectively.
Remark 4.15. Note the following more verbose formulation of the condition for a strategy $S$ for $\text{Reg}$ (for $\text{Reg}^+$) to be called a scope-delimiting (scope$^*$-delimiting) strategy:

— every source of a step is one of three kinds: the source of a $\rightarrow_{\lambda}$-step, the source of a $\rightarrow_{\text{del}}$-step (a $\rightarrow_S$-step), or the source of both a $\rightarrow_{@0}$-step and a $\rightarrow_{@1}$-step with the restriction that (in all three cases) it is not the source of any other step.

Mindful of the fact that sources of $\rightarrow_{\lambda}$-steps are not sources of $\rightarrow_{@i}$-steps, or vice versa, in $\text{Reg}$ (in $\text{Reg}^+$), this condition can be relaxed to the equivalent formulation:

— no source of a $\rightarrow_{\lambda}$-step or a $\rightarrow_{@i}$-step for $i \in \{0, 1\}$ is also the source of a $\rightarrow_{\text{del}}$-step (a $\rightarrow_S$-step), and every source of a $\rightarrow_{@i}$-step for $i \in \{0, 1\}$ is the source of both a $\rightarrow_{@0}$- and a $\rightarrow_{@1}$-step, but not the source of any other step.

Definition 4.16 (eager and lazy scope/scope$^*$-delimiting strategies). The eager scope-delimiting strategy $S_{\text{eag}}$ (lazy scope-delimiting strategy $S_{\text{lasy}}$) for $\text{Reg}$ is defined as the restriction of rewrite steps in $\text{Reg}$ to eager (and respectively, to lazy) application of the rule $\varrho_{\text{del}}$: on a term $M \in \text{Ter}(\lambda)$, applications of other rules are only allowed if $\varrho_{\text{del}}$ is not applicable (applications of $\varrho_{\text{del}}$ are only allowed when other rules are not applicable). Analogously, the eager scope$^*$-delimiting strategy $S'_{\text{eag}}$ (the lazy scope$^*$-delimiting strategy $S'_{\text{lasy}}$) for $\text{Reg}^+$ is defined as the restriction of rewrite steps in $\text{Reg}^+$ to eager (respectively, to lazy) application of the rule $\varrho^S$.

Remark 4.17 (history-aware versus history-free scope-delimiting strategies). The history-free strategy obtained by projection from a history-aware scope-delimiting strategy is not in general a scope-delimiting strategy. This is due to the non-determinism which may be introduced by the projection. Consider for example the term $MM$ with
$M = \lambda x.\lambda y. x \times y$ and the history-aware strategy $S$ constructed by using $S_{\text{eag}}$ on the left component and $S_{\text{lasy}}$ on the right component of $M M$. The sub-ARS induced by $MM$ then corresponds to the graphs of the sub-ARSs for $S_{\text{eag}}$ and $S_{\text{lasy}}$ as depicted in Figure 7 placed side by side with an additional connecting node at the top (also some leaves are merged). The induced sub-ARSs of the history-free strategy obtained by projection, however, resembles the graph from Figure 6 (with one additional application node at the top). That graph however bears the non-determinism which is not permitted for a scope-delimiting strategy, in the form of the existence of a source of both a $\to_s$ and a $\to_{s_{\text{reg}}}$ step.

The following proposition formulates a property of the eager scope-delimiting (scope$^*$-delimiting) strategy in $\text{Reg}$ (in $\text{Reg}^*$) that assigns it a special status: the target of every rewrite sequence with respect to $S_{\text{eag}}$ (with respect to $S_{\text{eag}}^*$) can be reached, modulo some final $\to_{\text{del}}$-steps ($\to_s$-steps), also by a rewrite sequence with respect to an arbitrary scope-delimiting (scope$^*$-delimiting) strategy. Furthermore, rewrite sequences with respect to $S_{\text{eag}}$ (with respect to $S_{\text{eag}}^*$) are able to mimic rewrite sequences with respect to an arbitrary scope-delimiting (scope$^*$-delimiting) strategy, up to trailing $\to_{\text{del}}$-steps ($\to_s$-steps) applied to the latter.

**Proposition 4.18.** For all scope-delimiting strategies $S$ on $\text{Reg}$, and scope$^*$-delimiting strategies $S^*$ on $\text{Reg}^*$ the following statements hold:

(i) $\to_{S_{\text{eag}}}$ factors into $\to_S$ and $\to_{\text{del}}$, and $\to_{S_{\text{eag}}^*}$ factors into $\to_{S^*}$ and $\to_s$.

(ii) $\to_{S_{\text{eag}}}$ is cofinal for $\to_S$ with trailing $\to_{\text{del}}$-steps, and $\to_{S_{\text{eag}}^*}$ is cofinal for $\to_{S^*}$ with trailing $\to_s$-steps.

**Definition 4.19 (generated subterms of $\lambda^\infty$-terms).** Let $S$ be a scope-delimiting strategy for $\text{Reg}$ with rewrite relation $\to_S$ (let $S^*$ be a scope-delimiting strategy for $\text{Reg}^*$ with rewrite relation $\to_{S^*}$). For every $M \in \text{Ter}(\lambda^\infty)$, the set $ST_S(M)$ of generated subterms of $M$ with respect to $\text{Reg}$ and $S$ (the set $ST^*_S(M)$ with respect to $\text{Reg}^*$ and $S$) is defined as the set of $\to_S$-reducts (the set of $\to_{S^*}$-reducts) of $M$ via the mappings:

$$ST_S : \text{Ter}(\lambda^\infty) \rightarrow \mathcal{P} (\text{Ter}(\lambda^\infty)) \hspace{1cm} ST^*_S : \text{Ter}(\lambda^\infty) \rightarrow \mathcal{P} (\text{Ter}(\lambda^\infty))$$

$$M \mapsto ST_S(M) := (M \to_S) \hspace{1cm} M \mapsto ST^*_S(M) := (M \to_{S^*})$$

**Definition 4.20.** Let $\mathcal{A}$ be an abstract rewriting system, and $S$ a strategy for $\mathcal{A}$. We say that an object $a$ in $\mathcal{A}$ is $S$-regular in $\mathcal{A}$ if the ARS $(a \to_S)$ induced by $a$ (and consequently the set of $\to_S$-reducts of $a$ in $\mathcal{A}$) is finite.

**Definition 4.21 (regular and strongly regular infinite $\lambda$-terms).** An infinite $\lambda$-term $M$ is called regular (strongly regular) if there exists a scope-delimiting strategy $S$ for $\text{Reg}$ (a scope$^*$-delimiting strategy $S^*$ for $\text{Reg}^*$) such that $M$ is $S$-regular (is $S^*$-regular).

Note that an infinite $\lambda$-term $M$ is regular (strongly regular) if and only if the set $ST_S(M)$ (the set $ST^*_S(M)$) of generated subterms of $M$ with respect to some scope-delimiting strategy $S$ (with respect to some scope$^*$-delimiting strategy $S^*$) is finite.
Proposition 4.22. The following statements hold:

(i) Every strongly regular infinite \( \lambda \)-term is also regular.

(ii) Finite \( \lambda \)-terms are both regular and strongly regular.

Proof. For statement (i), let \( M \) be an infinite \( \lambda \)-term, and let \( S \) be a scope-delimiting strategy for Reg* such that \( ST^*_S(M) \) is finite. Due to Proposition 4.4, \( S \) can be modified with the result of a scope-delimiting strategy \( S' \) for Reg by exchanging \( \rightarrow_S \)-steps with \( \rightarrow_{del} \)-steps. Then there is a stepwise correspondence between \( \rightarrow_S \)-rewrite sequences and \( \rightarrow_{del} \)-rewrite sequences that pass through the same terms. Consequently, the sets of \( \rightarrow_S \)- and \( \rightarrow_{del} \)-reducts of \( ()M \) coincide: \( ST^*_S(M) = ST^*_S(M) \). It follows that \( ST^*_S(M) \) is finite.
For statement (ii), note that by Proposition 4.8, (vi), and König’s Lemma, every finite term in \( \text{Ter}(\lambda(\lambda)) \) has only finitely many reducts with respect to \( \Rightarrow_{\text{reg}} \) or \( \Rightarrow_{\text{reg}^*} \). It follows that for every finite \( \lambda \)-term \( M \) and every scope-delimiting strategy \( S \) on \( \text{Reg} \) or on \( \text{Reg}^* \), the number of \( \Rightarrow_{S} \)-reducts of \( (\lambda)M \) is finite, too.

The scope-delimiting strategies in the above definition could be fixed to the respective eager versions without changing the notions of regular and strongly regular infinite \( \lambda \)-term.

**Proposition 4.23.** For all infinite \( \lambda \)-terms \( M \) the following statements hold:

(i) \( M \) is regular if and only if \( M \) is \( S_{\text{eag}} \)-regular.

(ii) \( M \) is strongly regular if and only if \( M \) is \( S_{\text{eag}^*} \)-regular.

**Proof.** We only prove (i), because (ii) can be established analogously. The implication “\( \Rightarrow \)” in (i) follows directly from the definition of regularity. For showing “\( \Leftarrow \)”, let \( M \) be an infinite \( \lambda \)-term that is regular. Then there exists a scope-delimiting strategy \( S \) so that \( ST_{S}(M) \) is finite. Since by Proposition 4.18, (i), every \( \Rightarrow_{S_{\text{eag}}} \)-rewrite sequence factors into an \( (\Rightarrow_{S_{\text{eag}}} \cdot \Rightarrow_{\text{del}}) \)-rewrite sequence, it follows that every term in \( ST_{S_{\text{eag}}}(M) \) is the \( \Rightarrow_{\text{del}} \)-reduct of a term in \( ST_{S}(M) \). As every term in \( \text{Ter}(\lambda^\omega) \) has only finitely many \( \Rightarrow_{\text{del}} \)-reducts, it follows that also \( ST_{S_{\text{eag}}}(M) \) is finite.

**Example 4.24 (regular and strongly regular terms).** The following examples demonstrate the connection between (strong) regularity and, as illustrated in Figure 8, the finiteness of the ARSs induced by \( \text{Reg} \) (\( \text{Reg}^* \)) strategies.

— Example 1.1 is regular but not strongly regular.

— Example 3.18 is strongly regular.

For further illustration of the statements made in Example 4.24 let us consider various \( \text{Reg} \) and \( \text{Reg}^* \) rewrite sequences corresponding to infinite paths through the terms.

**Example 4.25.** For the term \( M \) from Example 1.1, however we first introduce a finite CRS-based notation, as a 'higher-order recursive program scheme'. We can represent \( M \) by \( \lambda a.\text{rec}_M(a) \) together with the CRS-rule \( \text{rec}_M(X) \Rightarrow_{\text{rec}_M} \lambda x.\text{rec}_M(x) \cdot X \). It holds that \( \lambda a.\text{rec}_M(a) \Rightarrow_{\text{rec}_M} M \). Using this notation we can finitely describe the infinite path down the spine of Example 1.1 by the cyclic \( S_{\text{eag}} \) rewrite sequence:

\[
(\lambda a.\text{rec}_M(a) \Rightarrow_{\text{rec}_M} (\lambda a)\text{rec}_M(a) \Rightarrow_{\text{rec}_M} (\lambda ab)\text{rec}_M(b) a \Rightarrow_{\text{rec}_M} (\lambda b)\text{rec}_M(b) a \Rightarrow_{\text{del}} (\lambda b)\text{rec}_M(b) \Rightarrow_{\text{del}} (\lambda a)\text{rec}_M(a) \Rightarrow_{\text{rec}_M} (\lambda a.\text{rec}_M(a))
\]

In \( \text{Reg}^* \) the rewriting sequence for the same path is invariant over all scope\(^*\)-delimiting
strategies and necessarily infinite:

\[
\begin{align*}
(\lambda a) \rec_M(a) & \red{\Rightarrow} (\lambda a) \rec_M(a) & \rightarrow_{\text{reg}} \\
(\lambda a) \lambda b. \rec_M(b) a & \red{\Rightarrow} (\lambda b) \rec_M(b) a & \rightarrow_{\text{reg}} \\
(\lambda b) \lambda c. \rec_M(c) b & \red{\Rightarrow} (\lambda b) \rec_M(c) b & \rightarrow_{\text{reg}} \\
(\lambda b c) \lambda d. \rec_M(d) c & \red{\Rightarrow} (\lambda b c d) \rec_M(d) c & \rightarrow_{\text{reg}} \\
(\lambda b c d) \lambda e. \rec_M(e) c & \red{\Rightarrow} (\lambda b c d e) \rec_M(e) c & \rightarrow_{\text{reg}}
\end{align*}
\]

**Example 4.26.** As an illustration of Example 3.18 we study a slightly modified version \(M\) of the term, defined as the unfolding of \(\text{letrec } f = \lambda x. f \ y \ x\) in \(f\). It is strongly regular since the infinite path through the term can be witnessed by this cyclic \(\text{Reg}^*\) rewriting sequence:

\[
\begin{align*}
() \ M & = \\
() \ \lambda x y. M \ y \ x & \red{\Rightarrow} \\
(\lambda x) \ \lambda y. M \ y \ x & \red{\Rightarrow} \\
(\lambda x y) \ M \ y \ x & \rightarrow_{\beta_0} \\
(\lambda x y) \ M \ y & \rightarrow_{\beta_0} \\
(\lambda x) \ M & \rightarrow_5 \\
() \ M & \ldots
\end{align*}
\]

See also Figure 9 for a graphical illustration of the induced sub-ARS.

**Remark 4.27.** The restriction of Proposition 4.23 to the eager scope-delimiting strategy cannot be relaxed to arbitrary scope-delimiting strategies. The term in Example 4.25 for instance is \(S^\text{eag}\)-regular but not \(S^\text{hasy}\)-regular (Figure 9).

**Definition 4.28 (grounded cycles in \(\text{Reg}, \text{Reg}^*\)).** Let \(\tau : (\lambda \vec{x})_0 \ M_0 \rightarrow (\lambda \vec{x}_1) \ M_1 \rightarrow \ldots\) be a finite or infinite rewrite sequence with respect to \(\rightarrow_{\text{reg}}\) or \(\rightarrow_{\text{reg}^*}\). By a **grounded cycle** in \(\tau\) we mean a cycle \((\lambda \vec{x}_i) \ M_i \rightarrow (\lambda \vec{x}_{i+1}) \ M_{i+1} \rightarrow \ldots \rightarrow (\lambda \vec{x}_{i+k}) \ M_{i+k} = (\lambda \vec{x}_i) \ M_i\) in \(\tau\), where \(i \in \mathbb{N}\) and \(k \geq 1\), with the additional property that \(|\vec{x}_{i+j}| \geq |\vec{x}_i|\) for all \(j \in \{0, 1, \ldots, k\}\) (i.e. the lengths of the abstraction prefixes in the terms of the cycle is greater or equal to the length of the abstraction prefix at the first and final term of the cycle).

**Proposition 4.29.** Let \(M\) be an infinite \(\lambda\)-term that is \(S\)-regular (\(S^*\)-regular) for some scope-delimiting strategy \(S\) (scope*-delimiting strategy \(S^*\)). Then every infinite rewrite sequence with respect to \(S\) (with respect to \(S^*\)) contains a grounded cycle.

**Proof.** Since the argument is analogous in both cases, we only treat strongly regular terms. Let \(M\) be an infinite \(\lambda\)-term that is \(S^*\)-regular for some scope*-delimiting strategy \(S^*\), and let \(\tau : M = (\lambda \vec{x}_0) \ M_0 \rightarrow_{\text{reg}} (\lambda \vec{x}_1) \ M_1 \rightarrow_{\text{reg}} \ldots\) be an infinite rewrite sequence. As \(M\) is \(S^*\)-regular, the sequence \(\{(\lambda \vec{x}_i) \ M_i\}_{i \in \mathbb{N}}\) of terms on \(\tau\) contains only finitely many different terms. Let \(l := \lim \inf\{|\vec{x}_i|\}_{i \in \mathbb{N}}\), that is, the minimum of abstraction prefix lengths that appears infinitely often on \(\tau\). Let \(\{(\lambda \vec{x}_i) \ M_i\}_{j \in \mathbb{N}}\) be the subsequence of \(\{(\lambda \vec{x}_i) \ M_i\}_{i \in \mathbb{N}}\) consisting of terms with prefix length \(l\), and such that, for all \(k \geq i_0\), \(|\vec{x}_k| \geq l\). Since also this subsequence contains only finitely many terms, there exist \(j_1, j_2 \in \mathbb{N}, j_1 < j_2\)
such that $(\lambda \bar{x}_{i_1}) M_{i_1} = (\lambda \bar{x}_{i_2}) M_{i_2}$. By the choice of the subsequence it follows that $(\lambda \bar{x}_{i_1}) M_{i_1} \rightarrow_{S^*} \ldots \rightarrow_{S^*} (\lambda \bar{x}_{i_2}) M_{i_2}$ is a grounded cycle in $\tau$.

Rounding off this section, we describe a motivation for the system $Reg^+$ in terms of a operation ‘parse’ that at the same time (i) stepwisely decomposes infinite $\lambda$-terms into its generated subterms, and (ii) stepwisely recombines the decomposition analysis of generated subterms encountered with the original term as the result in the limit. With this purpose in mind, we define, below, a CRS $Parse^+$.

**Definition 4.30 ($Parse^+$).** Let $\Sigma_{(\lambda), parse^*} = \Sigma_{(\lambda)} \cup \{ parse^*_n | n \in \mathbb{N} \}$ be the extension of the CRS-signature for $(\lambda)$, where for $n \in \mathbb{N}$, the symbols $parse^*_n$ have arity $n$. By $Parse^+$
we denote the CRS with the following rules:

\[
\begin{align*}
(\rho^n_{\text{parse}}) & : \text{parse}^n(\tilde{X}_n, \text{pre}_n([\tilde{x}_n]) \text{app}(Z_0(\tilde{x}_n), Z_1(\tilde{x}_n))) \rightarrow \\
(\rho^\lambda_{\text{parse}}) & : \text{parse}^\lambda_n(\tilde{X}_n, \text{pre}_n([\tilde{x}_n]) Z_0(\tilde{x}_n)), \text{parse}^\lambda_n(\tilde{X}_n, \text{pre}_n([\tilde{z}_n] Z_1(\tilde{x}_n))) \\
(\rho^5_{\text{parse}}) & : \text{parse}_{n+1}^5(X_1, \ldots, X_{n+1}, \text{pre}_{n+1}([\tilde{x}_{n+1}] Z(\tilde{x}_n))) ightarrow \\
(\rho^0_{\text{parse}}) & : \text{parse}_0^n(X_1, \ldots, X_n, \text{pre}_n([\tilde{x}_n] Z(\tilde{x}_n))) ightarrow X_n
\end{align*}
\]

We denote by $\rightarrow_{\text{parse}}^*$ the rewrite relation induced by this CRS.

**Remark 4.31.** Observe that the rules $(\rho^i)$ for $i \in \{0, 1\}$, $(\rho^\lambda)$, and $(\rho^5)$ of the CRS $\text{Reg}^*$ are contained within the rules $(\rho^n_{\text{parse}})$, $(\rho^\lambda_{\text{parse}})$, $(\rho^5_{\text{parse}})$, respectively, of the CRS $\text{Parse}^*$, in the sense that applications of the latter rules include applications of the former.

This has as a consequence that repeated $\rightarrow_{\text{parse}}^*$-steps on a term $M$ lead to terms that contain generated subterms of $M$ as closed subexpressions. Furthermore $\rightarrow_{\text{parse}}^*$-rewrite sequences on $M$ are possible that move redexes simultaneously deeper and deeper, analysing ever larger parts of $M$, and at the same time recreating a larger and larger $\lambda$-term parts (stable prefix contexts) of $M$, the original term.

**Proposition 4.32.** For every term $M \in \text{Ter}(\lambda)$ it holds:

(i) $\text{Parse}^*$ analyses $M$ into its generated subterms: If $\text{parse}^n_{\text{parse}}(()) M \rightarrow_{\text{parse}}^*$ $M'$, then all subexpressions starting with $\text{pre}_n$ (for some $n \in \mathbb{N}$) in $M'$ are generated subterms of $M$.

Moreover, for every generated subterm $(\lambda\tilde{y}) N$ of $M$, there exists a $\rightarrow_{\text{parse}}^*$-reduction $M''$ of $M$ such that $(\lambda\tilde{y}) N$ is contained in $M''$.

(ii) $\text{Parse}^*$ reconstructs $M$: $\text{parse}^n_{\text{parse}}(()) M \rightarrow_{\text{parse}}^* M$.

**Example 4.33.** Let $M$ be the infinite unfolding of $\text{letrec} f = \lambda x y. f y x$ in $f$ for which we use as a finite representation the equation $M = \lambda x y. M y x$. In $\text{Parse}^*$, $M$ is decomposed, and composed again, by the infinite rewrite sequence (see also Example 3.12):

\[
\begin{align*}
\text{parse}^n_{\text{parse}}(()) M \\
= \text{parse}^n_{\text{parse}}(()) \lambda x. \lambda y. M y x \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \text{parse}^\ast_1(x', (\lambda x') \lambda y. M y x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_2(x', y', (\lambda x' y') M y' x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_3(x', y', (\lambda x' y') M y' x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_4(x', y', (\lambda x' y') M y' x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_5(x', y', (\lambda x' y') M y' x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_6(x', y', (\lambda x' y') M y' x') \\
\rightarrow_{\text{parse}}^\ast, \lambda \lambda x'. \lambda y'. \text{parse}^\ast_7(x', y', (\lambda x' y') M y' x') \\
\end{align*}
\]
\[
\gamma \gamma' \rightarrow_{\text{parse}^*} \lambda x'. \lambda y. \text{parse}_0^*((()) M) y' x' \\
= \lambda x'. \lambda y. \text{parse}_0^*((()) \lambda x. \lambda y. M \, y \, x) y' x'
\]

Note that the generated subterms of \( M \) appear as the last arguments of the functions \( \text{parse}_0^* \) in this rewrite sequence.

5. Observing \( \lambda_{\text{letrec}} \)-terms by their generated subterms

In this section we adapt the concepts developed so far for the infinitary \( \lambda \)-calculus to \( \lambda_{\text{letrec}} \). By combining the rules of \( R_\gamma \) with those of \( \text{Reg} \) and \( \text{Reg}^* \), respectively, we obtain the CRSs \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) for the deconstruction of \( \lambda_{\text{letrec}} \)-terms furnished with an abstraction prefix. We define scope-delimiting and scope\(^*-\)delimiting strategies for \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) as before by excluding all non-determinism except for sources of \( \rightarrow_{s_0} \) and \( \rightarrow_{s_1} \)-steps.

**Definition 5.1 (The CRSs \( \text{Reg} \) and \( \text{Reg}^* \) for decomposing \( \lambda_{\text{letrec}} \)-terms).** We extend \( \Sigma^{\lambda_{\text{letrec}}} \) (see Definition 3.3) by function symbols \( \text{pre}_n \) with arity one to obtain the signature \( \Sigma_{\lambda_{\text{letrec}}} = \Sigma_{\lambda_{\text{letrec}}} \cup \{ \text{pre}_n \mid n \in \mathbb{N} \} \) on which we define the \( \lambda_{\text{letrec}} \)-calculus with abstraction prefix. We denote the induced set of (finite) terms by \( \text{Ter}(\lambda_{\text{letrec}}) \) and we adopt the same informal notation for \( (\lambda_{\text{letrec}}) \) as for \( (\lambda^\omega) \) (see Definition 4.1).

On the signature \( \Sigma_{\lambda_{\text{letrec}}} \) we define the CRS \( \text{Reg}_{\text{letrec}} \) (the CRS \( \text{Reg}^*_{\text{letrec}} \)) with the rules as the union of the rules of \( R_\gamma \) and \( \text{Reg} \) (the union of the rules of \( R_\gamma \) and \( \text{Reg}^* \)).

**Definition 5.2 (The ARSs \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \)).** We denote by \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) the abstract rewriting systems (ARSs) induced by the CRSs \( \text{Reg}^*_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \), respectively. By \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) we denote the ARSs that result by restricting the ARSs \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \), respectively, to the subset \( \text{Ter}(\lambda_{\text{letrec}}) \) of terms.

Like in Definition 4.7 the symbols used for the rewrite relations of \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) are overloaded; we use the same symbols to denote the restrictions of said rewrite relations to \( \text{Ter}(\lambda_{\text{letrec}}) \).

The lemma below formulates a number of simple rewrite properties for \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \), mainly concerning the interplay between unfolding and decomposition steps.

**Lemma 5.3.** On \( \text{Ter}(\lambda_{\text{letrec}}) \), the rewrite relations in \( \text{Reg}_{\text{letrec}} \) and \( \text{Reg}^*_{\text{letrec}} \) have the following properties:

(i) \( \rightarrow_{\text{v}} \) one-step commutes with each of \( \rightarrow_{\lambda, \rightarrow_{s_0}}, \rightarrow_{s_1}, \rightarrow_s, \) and \( \rightarrow_{\text{del}} \):

\[
\left\{ \begin{array}{ll}
\leftrightarrow_{\text{v}} \cdot \rightarrow_{\lambda} \leq \rightarrow_{\lambda} \cdot \leftrightarrow_{\text{v}} & \\
\leftrightarrow_{\text{v}} \cdot \rightarrow_{s_0} \leq \rightarrow_{s_0} \cdot \leftrightarrow_{\text{v}} & \\
\leftrightarrow_{\text{v}} \cdot \rightarrow_{s_1} \leq \rightarrow_{s_1} \cdot \leftrightarrow_{\text{v}} & \\
\leftrightarrow_{\text{v}} \cdot \rightarrow_{\text{del}} \leq \rightarrow_{\text{del}} \cdot \leftrightarrow_{\text{v}} & \quad (i \in \{0, 1\})
\end{array} \right.
\]

(ii) \( \rightarrow_{\text{reg}} \) in \( \text{Reg}_{\text{letrec}} \) and \( \rightarrow_{\text{reg}^*} \) in \( \text{Reg}^*_{\text{letrec}} \) have the same normal forms as \( \rightarrow_{\text{reg}} \) in \( \text{Reg} \) and \( \rightarrow_{\text{reg}^*} \) in \( \text{Reg}^* \), respectively; \((\lambda x) \, x\) is the single term of \( \lambda_{\text{letrec}} \) in \( \rightarrow_{\text{reg}^*} \)-normal form.

Every \( \rightarrow_{\text{reg}^*} \)-normal form a term in \( \lambda_{\text{letrec}} \) is of the form \( (\lambda x_1 \ldots x_n) \, x_n \).
Proof. The commutation properties in (i) are easy to verify by analysing the behaviour of the rewrite rules in $Reg_{\text{letrec}}$ and in $Reg_{\text{letrec}}$ on the terms of $\text{Ter}(\lambda_{\text{letrec}})$.

The statement in (ii) follows from Proposition 4.8, (i), and (vi): Normal forms with respect to $\to_{\text{reg}}$ in $Reg_{\text{letrec}}$ and $\to_{\text{reg}}$ in $Reg_{\text{letrec}}$ can only be $\lambda$-terms without occurrences of $\text{letrec}$, since every occurrence of $\text{letrec}$ in a $\lambda_{\text{letrec}}$-term gives rise to a $\to_{\text{v}}$-redex.

Lemma 5.4. The rewrite relation $\to_{\text{v}}^\omega$ (to infinite $\to_{\text{v}}$-normal form in at most $\omega$ steps) one-step commutes with $\to_{\lambda}$, $\to_{\text{del}}$, $\to_{\text{letrec}}$, and $\to_{\text{del}}$:

$$\to_{\text{v}}^\omega \cdot \to_{\lambda} \subseteq \to_{\lambda} \cdot \to_{\text{v}}^\omega \quad \to_{\text{v}}^\omega \cdot \to_{\text{del}} \subseteq \to_{\text{del}} \cdot \to_{\text{v}}^\omega \quad (i \in \{0, 1\})$$

This implies, for prefixed terms that have unfoldings that:

$$U((\lambda x)(\lambda y) L_0) \to_{\lambda} U((\lambda x y) L_0) \quad \to_{\lambda} U((\lambda x) L_0 L_1) \to_{\lambda_1} U((\lambda x) L_i) \quad (i \in \{0, 1\})$$

$$(\lambda x_1 \ldots x_{n+1}) M \to_{\text{del}} (\lambda x_1 \ldots x_n) M \Rightarrow U((\lambda x_1 \ldots x_{n+1}) M) \to_{\text{del}} U((\lambda x_1 \ldots x_n) M)$$

Furthermore it holds: $\to_{\text{v}}^\omega \cdot \to_{\text{v}} \subseteq \to_{\text{v}}^\omega$.

Proof. The commutation properties with the rewrite relation $\to_{\text{v}}^\omega$ can be shown by using refined versions of the commutation properties in Lemma 5.3, (i), in which the minimal depth of unfolding steps is taken account of. When denoting by $\to_{\text{v}}^n$ the rewrite relation that is generated by $\to_{\text{v}}$-steps of depth $\geq n$, then the following properties hold:

$$\to_{\text{v}}^n \cdot \to_{\lambda} \subseteq \to_{\lambda} \cdot \to_{\text{v}}^n \quad \to_{\text{v}}^n \cdot \to_{\text{del}} \subseteq \to_{\text{del}} \cdot \to_{\text{v}}^n$$

Using these properties, strongly convergent $\to_{\text{v}}$-rewrite sequences can be shown to project, over $\to_{\lambda}$, $\to_{\text{del}}$, and $\to_{\text{letrec}}$-steps, to again strongly convergent $\to_{\text{v}}$-rewrite sequences.

The property $\to_{\text{v}}^\omega \cdot \to_{\text{v}} \subseteq \to_{\text{v}}^\omega$ can be shown by using refined versions of the elementary diagrams from the confluence proof that take the the minimal depths of steps into account.

As for $Reg$ ($Reg'$) a we require of scope-delimiting strategies to have deterministic $\to_{\text{del}}$-steps ($\to_{\text{del}}$-steps). As was the case for scope-delimiting and scope$^+$-delimiting strategies for $Reg$ and $Reg'$, these strategies will also here fix all non-determinism except for the choice between $\to_0$ and $\to_1$.

Definition 5.5 (scope/scope$^+$-delimiting strategy for $\lambda_{\text{letrec}}$-terms). A strategy $S$ for $Reg_{\text{letrec}}$ (for $Reg_{\text{letrec}}^+$) will be called a scope-delimiting (scope$^+$-delimiting) strategy if:

- $S$ is deterministic for sources of $\to_{\lambda}$-steps, $\to_{\text{del}}$-steps ($\to_{\text{del}}$-steps), and all $\text{letrec}$-unfolding steps (i.e. all $\to_{\text{S}}$-steps for every rule $\rho_{\text{S}}^\omega$ of $R_S$).

- $S$ enforces eager application of $\rho_{\text{red}}^\omega$: every source of a step in $S$ according to a rule different from $\rho_{\text{red}}^\omega$ is not the source of a $\to_{\text{red}}$-step.

We say that such a strategy $S$ is a lazy-unfolding scope-delimiting strategy (a lazy-unfolding scope$^+$-delimiting strategy) if furthermore:
— \(S\) applies the rules of \(R_{\varphi,\text{red}}\) only at the root of the term, i.e. directly beneath the abstraction prefix except for \(\varphi_{\text{red}}\).
— \(S\) uses the rules of \(R_{\varphi}\) other than reduction rule \(\varphi_{\text{red}}\) in a lazy way: for every source \(s\) of a step in \(S\) with respect to a rule of \(R_{\varphi}\) other than \(\varphi_{\text{red}}\) it holds that \(s\) is not also the source of a step, in the ARS underlying the strategy, with respect to one of the rules of \(\text{Reg (of } R_{\varphi}^*)\).

For every scope-delimiting strategy \(S\) on \(\text{Reg}_{\text{letrec}}\) (on \(\text{Reg}_{\text{letrec}}'\)), we denote by \(\rightarrow_{\text{a},S} , \rightarrow_{\text{r},S}, \rightarrow_{\text{S},S} , \rightarrow_{\text{del},S} ,\) and by \(\rightarrow_{\varphi,\text{red},S}, \rightarrow_{\varphi,\text{red},S} , \rightarrow_{\varphi,\text{red},S} , \rightarrow_{\varphi,\text{red},S} , \rightarrow_{\varphi,\text{red},S}\) the rewrite relations that are induced by those steps according to \(S\) that result by applications of the rules \(\varphi_{\text{del}}, \varphi_{\text{del}}, \varphi_{\text{del}}, \varphi_{\text{del}}\) of \(\text{Reg}\) and \(\text{Reg}^*\), and the rules \(\varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}, \varphi_{\varphi}\) of \(R_{\varphi}\), respectively.

**Remark 5.6.** We need to permit the application of \(\varphi_{\text{red}}\) anywhere inside the term to handle terms that contain inaccessible bindings in binding groups. Otherwise the possibility to apply \(\varphi_{\text{del}}(\varphi_{\text{del}})\) may be blocked in the case of a prefixed \(\lambda_{\text{letrec}}\)-term \((\lambda \bar{x})L\) in which a variable from the abstraction prefix is vacuous with respect to the infinite unfolding of \((\lambda \bar{x})L\) but is bound by a term in an inaccessible binding of some binding group in \(L\).

We can however restrict applicability of \(\varphi_{\text{red}}\) to outermost redexes if we start unfolding on reduced terms, that is, \(\rightarrow_{\varphi,\text{red}}\)-normal forms, because then \(\varphi_{\text{red}}\)-redexes can only arise at outermost positions (by application of the \(\varphi_{\text{a}}\)-rule).

**Remark 5.7 (non-deterministic unfolding).** Note that in Definition 5.5 we do not only require of a strategy to eliminate the non-determinism with respect to \(\varphi_{\text{del}}\)-transitions (\(\varphi_{\text{del}}\)-transition) but all non-determinism except for \(\varphi_{\text{del}}\) non-determinism. This restriction will play a role later for the definition of \(\lambda\)-transition graphs in Section 9, and here below for the definition of the projection of scope-delimiting (scope\(^*\)-delimiting) strategies on \(\lambda_{\text{letrec}}\)-terms to scope-delimiting (scope\(^*\)-delimiting) strategies on infinite \(\lambda\)-terms.

**Remark 5.8 (eager application of \(\varphi_{\text{red}}\)).** By requiring scope/scope\(^*\) delimiting strategies to apply \(\varphi_{\text{red}}\) eagerly we can exploit a useful property with respect to free variables of a term: if \((\lambda \bar{x})L \in \text{Ter}(\lambda_{\text{letrec}})\) is in \(\rightarrow_{\varphi,\text{red}}\)-normal form, then the free variables occurring in \(M\) correspond to the free variables of \(U(L)\).

Note that here we have applied \(U\) to a prefixed term which requires an extension of \(U\) to terms in \(\text{Ter}(\lambda_{\text{letrec}})\).

**Definition 5.9 (unfolding of prefixed terms).** We redefine \(U\) as a partial function over the domain \(\text{Ter}(\lambda_{\text{letrec}})\):

\[
U : \text{Ter}(\lambda_{\text{letrec}}) \rightarrow \text{Ter}(\lambda^\omega)
\]

\[L \mapsto M \quad \text{if } L \rightarrow^\omega_M\]

Uniqueness of \(U\) follows from Lemma 3.15.

**Definition 5.10 (productive terms w.r.t. scope-/scope\(^*\)-del. strat.).** Let \(L\) be a \(\lambda_{\text{letrec}}\)-term, and \(S\) a scope-delimiting strategy for \(\text{Reg}_{\text{letrec}}\) or a scope\(^*\)-delimiting strategy
for \( \text{Reg}_{\text{letrec}} \). We say that \( L \) is \( S \)-\textit{productive} if every infinite rewrite sequence on \( L \) with respect to \( S \) contains infinitely many steps according to \( \rightarrow_{\text{st},S} \), \( \rightarrow_{a_0,S} \), or \( \rightarrow_{\lambda,S} \).

**Definition 5.11** (generated subterms of \( \lambda_{\text{letrec}} \)-terms). Let \( S \) be a scope-delimiting strategy for \( \text{Reg}_{\text{letrec}} \) (for \( \text{Reg}^*_{\text{letrec}} \)). For every \( L \in \text{Ter}(\lambda_{\text{letrec}}) \), the set \( ST_S(L) \) (the set \( ST^+_S(L) \)) of generated subterms of \( L \) with respect to \( \text{Reg}_{\text{letrec}} \) and \( S \) (with respect to \( \text{Reg}^*_{\text{letrec}} \) and \( S \)) is defined as the set of \( \rightarrow_{\text{st}} \)-many-step reducts (the set of \( \rightarrow_{\text{st}} \)-many-step reducts) of \( ()L \) via the mappings:

\[
ST_S: \text{Ter}(\lambda_{\text{letrec}}) \rightarrow \mathcal{P}(\text{Ter}(\lambda_{\text{letrec}})) \quad \text{and} \quad ST^+_S: \text{Ter}(\lambda_{\text{letrec}}) \rightarrow \mathcal{P}(\text{Ter}(\lambda_{\text{letrec}}))
\]

\( L \mapsto ST_S(L) := ((()L) \rightarrow_S) \quad \text{and} \quad L \mapsto ST^+_S(L) := ((()L) \rightarrow^+_S) \)

The following lemma states that every scope\(^*\)-delimiting strategy for \( \text{Reg}^*_{\text{letrec}} \) (on \( \lambda_{\text{letrec}} \)-terms), when restricted to the reducts of a \( \lambda_{\text{letrec}} \)-term \( L \) that expresses an infinite \( \lambda \)-term \( M \), projects to the restriction of a scope\(^*\)-delimiting strategy for \( \text{Reg}^* \) (on infinite \( \lambda \)-terms) to reducts of \( M \). And it asserts a similar statement for scope-delimiting strategies. The projection hereby makes use of the commutation properties described in Lemma 5.4 between the infinite unfolding \( \rightarrow^*_\nu \) and the decomposition rewrite relations \( \rightarrow_{\lambda}, \rightarrow_{a_0}, \rightarrow_{\lambda_\nu}, \rightarrow_{a_1}, \rightarrow_S \), and \( \rightarrow^* \).

**Lemma 5.12.** Let \( S \) be a scope\(^*\)-delimiting strategy \( S \) for \( \text{Reg}^*_{\text{letrec}} \) (for \( \text{Reg}^*_{\text{letrec}} \)), and let \( L \) be a \( \lambda_{\text{letrec}} \)-term that is \( S \)-productive. Then there exists a (history-aware) scope\(^*\)-delimiting strategy (scope\(^*\)-delimiting strategy) \( S \) for \( \text{Reg} \) (for \( \text{Reg}^* \)) such that the induced sub-ARS \( (U(L) \rightarrow_S) \) of \( U(L) \) is the projection (under the unfolding mapping \( U \)) of the induced sub-ARS \( (L \rightarrow^*_S) \) of \( L \), in the sense that for all \( L' \) in \( (L \rightarrow_S) \) it holds:

\[
L' \rightarrow^*_{\nu,S} \rightarrow_{\lambda/a_0/S/\text{del},S} L'' \quad \implies \quad U(L') \rightarrow_S U(L'')
\]

\[
U(L') \rightarrow_S M'' \quad \implies \quad (3L'') [ L' \rightarrow^*_{\nu,S} \rightarrow_{\lambda/a_0/S/\text{del},S} L'' \wedge M'' = U(L'') ]
\]

As a consequence, \( U(L) \) is \( S \)-regular if \( L \) is \( S \)-regular.

**Proof sketch** We can utilise Lemma 5.4 to make commuting diagrams out of the two formulas above (for any given \( L' \)), which allows us to determine \( S \) with respect to all terms in \( (L \rightarrow_S) \). This freedom in the definition of \( S \) also guarantees the property in the second implication in the lemma. \( \square \)

**Definition 5.13** (\( \text{Parse}^*_\nu \)). By \( \text{Parse}^*_\nu \) we denote the CRS comprising the rules of \( \text{Parse}^*_\nu \) as well as the rules from \( R_\nu \).

**Example 5.14.** When applied to \( \text{letrec} f = \lambda xy. f \ y \ x \ in \ f \) is \( \text{Parse}^*_\nu \) unfolds and decomposes, but at the same time recreates the corresponding infinite \( \lambda \)-term (see also Example 3.12 and Example 4.33):

\[
\text{parse}^*_\nu((() \text{letrec} f = \lambda xy. f \ y \ x \ in \ f))
\]

\[
\rightarrow_{\nu, \text{rec}} \text{parse}^*_\nu((() \text{letrec} f = \lambda xy. f \ y \ x \ in \ \lambda xy. f \ y \ x))
\]

\[
\rightarrow_{\nu, \lambda} \text{parse}^*_\nu((() \text{letrec} f = \lambda xy. f \ y \ x \ in \ \lambda y. f \ y \ x))
\]

\[
\rightarrow_{\text{parse}^* \lambda, \lambda} \lambda x. \text{parse}^*_\nu(x, (\lambda x) \text{letrec} f = \lambda xy. f \ y \ x \ in \ \lambda y. f \ y \ x)
\]
Lemma 5.15. For all closed \( L \in \text{Ter}(\lambda_{\text{letrec}}) \) the following statements are equivalent:

(i) \( L \) expresses an infinite \( \lambda \)-term \( M \), that is, \( L \rightarrow^{*} M \).

(ii) \( \text{parse}^{0}(()) \rightarrow^{\omega}_{\text{par}} M \), for some infinite \( \lambda \)-term \( M \).

(iii) \( L \) is \( S^{*} \)-productive for some scope* -delimiting strategy \( S^{*} \).

(iv) \( L \) is \( S^{*} \)-productive for every scope* -delimiting strategy \( S^{*} \).

Proof. Let \( L \in \text{Ter}(\lambda_{\text{letrec}}) \). We show the lemma by establishing the implications in the following order: “(iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i) \( \Rightarrow \) (iv)”.

The implication “(iv) \( \Rightarrow \) (iii)” is clear: (iv) implies that \( L \) is productive for e.g. the lazy-unfolding, eager scope* -delimiting strategy on \( \text{Reg}^{0}_{\text{letrec}} \).

For showing the implication “(iii) \( \Rightarrow \) (ii)”, let \( S^{*} \) be a scope* -delimiting strategy for \( \text{Reg}^{*} \) such that \( L \) is \( S^{*} \)-productive. Then the strategy \( S^{*} \) defines a \( \rightarrow_{\text{par}} \)-rewrite sequence \( \tau \) on \( \text{parse}^{0}(()L) \) by using \( S^{*} \) to define next steps on subexpressions that are of the form \( \text{parse}^{0}_{n}(..., \text{pre}_{n}([x_{1}, \ldots, x_{n}] P)) \) in already obtained reducts: if on a term \( (\lambda x_{1}, \ldots, x_{n}) P \) the strategy \( S^{*} \) prescribes a \( \rightarrow_{\text{par}} \)-step, then this step is taken over in \( \tau \); if \( S^{*} \) prescribes a \( \rightarrow_{\lambda_{\text{eq}}} \)-step, then \( \tau \) can continue with a \( \rightarrow_{\text{par}} \) -\( \lambda \)-step; if \( S^{*} \) prescribes a \( \rightarrow_{\lambda_{\text{rec}}} \)-step and a \( \rightarrow_{\lambda_{\text{rec}}} \) -\( \lambda \)-step, then \( \tau \) can continue with a \( \rightarrow_{\text{par}} \) -\( \lambda \)-step. For the construction of \( \tau \), possible steps in subexpressions \( \text{parse}^{0}_{n}(..., \text{pre}_{n}([x_{1}, \ldots, x_{n}] P)) \) at parallel positions have to be interleaved to ensure that the reduction work is done in an outermost-fair way. Productivity of \( S^{*} \) on \( L \) ensures that always after finitely many steps inside a subexpression \( \text{parse}^{0}_{n}(..., \text{pre}_{n}([x_{1}, \ldots, x_{n}] P)) \) the function symbol \( \text{parse}^{0}_{n} \) disappears at this position (either entirely, or it is moved deeper over a \( \lambda \)-abstraction or an application). In the terms of the rewrite sequence \( \tau \) larger and larger \( \lambda \)-term contexts appear at the
6. Proving regularity and strong regularity

In this section we introduce proof systems that are sound and complete for the notions of regular, and strongly regular, infinite \( \lambda \)-terms. In order to prove soundness and completeness, we establish, as auxiliary results, a correspondence between scope-delimiting/\( \text{scope}^* \)-delimiting strategies for \( \text{Reg/Reg}^* \) and closed derivations in the corresponding proof systems. Then we introduce a proof system that is sound and complete for equality between strongly regular infinite \( \lambda \)-terms. Furthermore, we give two proof systems that are sound and complete for the property of \( \lambda_{\text{letrec}} \)-terms to unfold to infinite \( \lambda \)-terms. And finally, we show the following part of our characterisation result: infinite \( \lambda \)-terms that are unfoldings of \( \lambda_{\text{letrec}} \)-terms are strongly regular.

We start with a more formal definition of \( \lambda \)-terms and \( \lambda_{\text{letrec}} \)-terms than in Definition 3.3, by means of derivability in proof system that formalises term decomposition.

**Definition 6.1 (infinite \( \lambda \)-terms).** We define the set of prefixed infinite \( \lambda \)-terms as those terms in \( \text{Ter} (\Sigma (\lambda)) \) for which there exists a possibly infinite, completed (see Definition 2.3)
\[
\begin{array}{c}
\frac{(\lambda x y) y}{0} \\
\frac{(\lambda x y) M_0}{(\lambda x) y M_0} \lambda \\
\frac{(\lambda x) M_0}{(\lambda x) M_0 M_1} @ \\
\frac{(\lambda x_1 \ldots x_{n-1}) M}{(\lambda x_1 \ldots x_n) M} \ S \text{ (if the binding } \lambda x_n \text{ is vacuous)}
\end{array}
\]

Fig. 10. Proof system \(T^\infty_\lambda\) for defining the set of infinite \(\lambda\)-terms.

\[
\begin{array}{c}
\frac{(\lambda f_1 \ldots f_n) M_1 \ldots (\lambda f_0 \ldots f_n) M_n}{(\lambda f) \text{letrec } f_1 = M_1, \ldots, f_n = M_n \text{ in } M}
\end{array}
\]

Fig. 11. Proof system \(T_{(\text{letrec})}\) defined as an extension of \(T^\infty_\lambda\) by an additional rule to define the set of \(\lambda\text{letrec}\)-terms.

derivation in the proof system \(T^\infty_\lambda\) with axioms and rules as shown in Figure 10:

\[\text{Ter}((\lambda^\infty)) := \{ M \in \text{Ter}(\Sigma_\lambda) \mid \vdash_{T^\infty_\lambda} M \}\]

The set of plain infinite \(\lambda\)-terms are those terms that comply with the previous definition when equipped with an empty prefix:

\[\text{Ter}(\lambda^\infty) := \{ M \in \text{Ter}(\Sigma_\lambda) \mid \vdash_{T^\infty_\lambda} () M \in \text{Ter}((\lambda^\infty)) \}\]

**Definition 6.2 (\(\lambda\text{letrec}\)-terms).** The set of prefixed \(\lambda\text{letrec}\)-terms comprises those terms out of \(\text{Ter}(\Sigma_{(\text{letrec})})\) for which there exists a finite derivation in the proof system \(T_{(\text{letrec})}\) (Figure 11):

\[\text{Ter}(\lambda_{\text{letrec}}) := \{ M \in \text{Ter}(\Sigma_{(\text{letrec})}) \mid \vdash_{T_{(\text{letrec})}} M \}\]

The set of plain \(\lambda\text{letrec}\)-terms are those terms that comply with the previous definition when equipped with an empty prefix:

\[\text{Ter}(\lambda_{\text{letrec}}) := \{ M \in \text{Ter}(\Sigma_{\lambda_{\text{letrec}}}) \mid \vdash_{T_{(\text{letrec})}} () M \in \text{Ter}((\lambda_{\text{letrec}})) \}\]

Building on rules already used in the proof systems for term formation in \(\lambda^\infty\) and \((\lambda^\infty)\) from the definition above, we now introduce proof systems for regularity and strong regularity of infinite \(\lambda\)-terms in \(\lambda^\infty\).

**Definition 6.3 (proof systems Reg, and Reg\(^+\), Reg\(^+_0\)).** The natural-deduction style proof system \(\text{Reg}\(^+\)\) for recognising strongly regular, infinite \(\lambda\)-terms contains the axioms and rules as shown in Figure 12. In particular, the rule (FIX) is a natural-deduction style derivation rule in which marked assumptions from the top of the proof tree can be discharged. Instances of this rule carry the side-condition that the depth \(D_0\) of the immediate subderivation \(D_0\) of its premise is greater or equal to 1 (hence this subderivation contains at least one rule instance, and, importantly, for a topmost occurrence of (FIX), \(D_0\) must have a bottommost instance of one of the rules (\(\lambda\)), (\(\@\)), or (\(\text{S}\))).

The variant \(\text{Reg}\(^+_0\)\) of \(\text{Reg}\) contains the same axioms and rules as \(\text{Reg}\(^+\)\), but in it
Vacuous (if the binding ‘guardedness’ property for threads from such instances upwards to discharged instances.

(Fig. 13, and by the restriction of the axiom scheme (Vacuous) in Figure 13, and by the restriction of the axiom scheme (Vacuous) for regular infinite \( \lambda \)-terms arises from the proof system \( \text{Reg}^* \) by replacing the rule (S) with the rule (Vacuous) for the introduction of vacuous bindings in the \( \lambda \)-abstraction prefixes, and by replacing the axiom scheme (0) of \( \text{Reg}^* \) by the more restricted version here.

Fig. 12. The natural-deduction style proof system \( \text{Reg}_0^\oplus \) for strongly regular infinite \( \lambda \)-terms, which is an extension of \( T_{(\lambda)}^\oplus \) by one additional rule (FIX). In the variant system \( \text{Reg}_0^\oplus \), instances of (FIX) are subject to the following side-condition: for all \((\lambda y)N\) on threads in \( \mathcal{D}_0 \) from open marked assumptions \(((\lambda x)M)^u\) downwards it holds that \(|y| \geq |x|\).

Remark 6.4. The proof system \( \text{Reg}^* \) is related to a proof system for nameless, finite terms in the \( \lambda \)-calculus that is used in (van Oostrom et al., 2004, sec. 2) as part of a translation of \( \lambda \)-terms into ‘Lambdascope’ interaction nets, which are used for optimal evaluation (in the sense of Lévy) of \( \lambda \)-terms.

\begin{align*}
\frac{0}{(\lambda y)y} & \quad \frac{(\lambda x_1 \ldots x_{n-1})M}{(\lambda x_1 \ldots x_n)M} \quad \frac{(\lambda x_1 \ldots x_{n-1})M_0}{(\lambda x_1 \ldots x_n)M_0} \quad \frac{(\lambda x)M_0}{(\lambda x)M_1 \oplus} \\
\frac{(\lambda x_1 \ldots x_n)M}{(\lambda x_1 \ldots x_n)M} & \quad \frac{\lambda x_n}{\lambda x_n} \quad \frac{\lambda x_n}{\lambda x_n} \quad \frac{\lambda x_n}{\lambda x_n} \quad \frac{\lambda x_n}{\lambda x_n}
\end{align*}

The proposition below explains that the side-condition on instances of (FIX) from the proof systems above to have immediate subderivations \( \mathcal{D}_0 \) with \(|\mathcal{D}_0| \geq 1\) entails a ‘guardedness’ property for threads from such instances upwards to discharged instances.

Proposition 6.5. Let \( \mathcal{D} \) be a derivation in \( \text{Reg}_0^\oplus \) (or in \( \text{Reg}^* \) possibly with open marked assumptions. Then for all instances \( \iota \) of the rule (FIX) in \( \mathcal{D} \) it holds: every thread from \( \iota \) upwards to a marked assumption that is discharged at \( \iota \) passes at least one instance of a rule (\( \lambda \)) or (\( \oplus \)).
Proof. Since for \( \text{Reg}' \) and \( \text{Reg}_0' \) the argument is analogous, we only consider derivations in \( \text{Reg} \). So, let \( D \) be a derivation in \( \text{Reg} \). Furthermore, let \( \iota \) be an instance of the rule (\( \text{FIX} \)) in \( D \) with conclusion \((\lambda \vec{x})M\) and let \( \pi \) be a thread from the conclusion of \( \iota \) upwards to a marked assumption \((\lambda \vec{x})M\_u\). Let \( \kappa \) be the topmost instance of (\( \text{FIX} \)) in \( D \) that is passed on \( \pi \). By its side-condition, the immediate subderivation of \( \kappa \) has depth greater or equal to 1, and hence there is at least one instance of a rule (\( \lambda \)), (\( \& \)), or (\text{Vacuous}) passed on \( \pi \) above \( \kappa \). If there is an instance of (\( \lambda \)) or (\( \& \)) on this part of \( \pi \), we are done.

Otherwise only rules (\text{Vacuous}) are passed on \( \pi \) above \( \kappa \). But since the rule (\text{Vacuous}) decreases the length of the abstraction prefix in the term occurrences in a pass from the conclusion to the premise, and since the length of the abstraction prefix at the start of \( \pi \) is the same as the end of \( \pi \), namely \([\vec{x}]\), it follows that at least one occurrence of a rule that increases the length of the abstraction prefix must have been passed on \( \pi \), too, on the part from \( \iota \) to \( \kappa \). Since the only rule of \( \text{Reg} \) that increases the length of an abstraction prefix in a pass from conclusion to a premise is the rule (\( \lambda \)), we have also in this case found a desired rule instance on \( \pi \).

Example 6.6. Let \( M \) be the infinite unfolding of \text{letrec} \( f = \lambda xy. fx \) in \( f \) for which we use as a finite representation the equation \( M = \lambda xy. Mx \) This term admits the following derivations in \( \text{Reg}_0^* \), where as opposed to the former the latter has some redundancy:

\[
\begin{array}{c}
(\lambda x)M \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda xy)Mx \\
\downarrow \\
(\lambda xy)Mx \\
\downarrow \\
(\lambda xy)Mx \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda x)Mx \\
\downarrow \\
(\lambda x)Mx
\end{array}
\begin{array}{c}
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy \\
\downarrow \\
(\lambda x)\lambda y.Mxy
\end{array}
\]

Note that the derivation on the left is also a derivation in \( \text{Reg}_0^* \), but that this is not the case for the derivation on the right. There, for the occurrence of (\text{FIX}) the side-condition in the system \( \text{Reg}_0^* \) is violated: on the path from the marked assumption \((\lambda x)\lambda y.Mxy\) down to the instance of (\text{FIX}) there is the occurrence \((\lambda x)M\) of a term with shorter prefix than the term in the assumption and in the conclusion. See Example 8.5 for a comparison of annotated versions of the two derivations above.

See Example 4.26 for a rewriting sequence in \( \text{Reg}' \) corresponding to the leftmost path in both derivations, and also Figure 9 for the corresponding transition graph.

Remark 6.7 (\text{Reg}' versus \text{Reg}_0^*). While it will be established in Proposition 6.12 below that provability in \( \text{Reg}' \) and \( \text{Reg}_0^* \) coincides, the difference between these systems will come to the fore in annotated versions that are purpose-built for the extraction of \( \lambda \text{letrec} \)-terms that express infinite \( \lambda \)-terms. This will be explained and illustrated later in Example 8.5, using annotated versions of the two derivations in Example 6.6 above.

Example 6.8. The infinite \( \lambda \)-term from Example 1 with the \( \lambda \)-transition graph shown
in Figure 8 is derivable in \( \text{Reg} \) by the following closed derivation using the notation from Example 4.25:

\[
\begin{array}{ll}
(\lambda b).rec_M(b) \\
(\lambda a).rec_M(a)
\end{array}
\]

When trying to construct a derivation for this term in \( \text{Reg}^+ \) from the bottom upwards, the rules of \( \text{Reg}^+ \) apart from (in first instance) (\text{FIX}) offer only deterministic choices, with as outcome an infinite proof tree of the form:

\[
\vdots
\]

But then, since this proof tree does not contain repetitions, also use of the rule (\text{FIX}) in order to discharge assumptions is impossible. Consequently, the term is not derivable in \( \text{Reg}^+ \).

Proposition 6.9. Let \( D \) be a derivation in \( \text{Reg} \) (in \( \text{Reg}^+ \)) with conclusion \( (\lambda \bar{x})M \).

Then every path in \( D \) from the conclusion upwards corresponds to a \( \rightarrow_{\text{reg}} \)-rewrite sequence (a \( \rightarrow_{\text{reg}^+} \)-rewrite sequence) from \( (\lambda \bar{x})M \): while passes over instances of (\text{FIX}) correspond to empty steps, passes over instances of the rule (\text{Vacuous}) to the left and to the right correspond to \( \rightarrow_{\text{del}} \)-steps, respectively; passes over instances of the rules (\lambda) and (Vacuous) (the rule (S)) correspond to \( \rightarrow_{\lambda} \)- and \( \rightarrow_{\text{del}} \)-steps, respectively. The same holds for (finite or infinite) cyclic paths in \( D \) that return, possibly repeatedly, from a marked assumption at the top down to the conclusion of the instance of (\text{FIX}) at which the respective assumption is discharged.

Proof. The proposition is an easy consequence of the following facts: passes from a term in the conclusion of an instance \( \iota \) of one of the rules (\lambda), (Vacuous), (S) to the term the premise of \( \iota \) correspond to \( \rightarrow_{\lambda} \), \( \rightarrow_{\text{del}} \), and \( \rightarrow_{S} \)-steps, respectively; passes from a term
in the conclusion of an instance of (⃗⃗) to the left and the right premise correspond to →_a₀- and →_a₁-steps, respectively.

Observe that, for the derivation \( D \) in Example 6.6, the →_{reg⁺}-rewrite sequences that correspond to paths in \( D \) as described in Proposition 6.9 are actually rewrite sequences with respect to the eager scope⁺-delimiting strategy \( S^{+}_{\text{eag}} \) for \( \text{Reg}^⁺ \). This illustrates the general situation, formulated by the lemma below: paths in a derivation \( D \) in \( \text{Reg}^⁺ \) (or in \( \text{Reg} \)) from the conclusion upwards correspond to rewrite sequences according to some, usually history-aware, scope⁺-delimiting (scope-delimiting) strategy \( S \) for \( \text{Reg}^⁺ \) (for \( \text{Reg} \)), which can be extracted from \( D \).

**Lemma 6.10 (from Reg/Reg⁺-derivations to scope/scope⁺-delim. strategies).**

Let \( M \in \text{Ter}(\mathcal{X}^⁺) \), and let \( D \) be a closed derivation in \( \text{Reg} \) (in \( \text{Reg}^⁺ \)) with conclusion \((M)\). Then there exists an, in general history-aware, scope-delimiting strategy \( S_D \) for \( \text{Reg} \) (scope⁺-delimiting strategy \( S_{\text{reg}^⁺} \) for \( \text{Reg}^⁺ \)) with the following properties:

(i) Every (possibly cyclic) path in \( D \) from the conclusion upwards corresponds to a rewrite sequence with respect to \( S_D \) starting on \((M)\) in the sense of Proposition 6.9 where passes over instances of the rules (⃗⃗) to the left and to the right correspond to →_a₀,ₜₚ- and →_a₁,ₜₚ-steps, respectively, and passes over instances of (\( λ \)) and of (Vacuous) (of (S)) correspond to →_λ,ₜₚ- and →_delₜₚ-steps (→ₜₚ-steps).

(ii) Every rewrite sequence that starts on \((M)\) and proceeds according to \( S_D \) corresponds to a (possibly cyclic) path in \( D \) starting at the conclusion in upwards direction: thereby a →_a₀,ₜₚ- and →_a₁,ₜₚ-step corresponds to a pass over (possibly successive (FIX)-instances, or from a marked assumption to the instance of (FIX) that binds it, followed by) an instance of (⃗⃗) in direction left and right, respectively; a →_λ,ₜₚ- or →_delₜₚ-step (→ₜₚ-step) corresponds to a pass over (possibly (FIX)-instances and assumption bindings to (FIX)-instances) an instance of (\( λ \)) or (Vacuous) (of (S)), respectively.

(iii) \( ST_{ₜₚ}(M) = \{ (λy)N | \text{the term } (λy)N \text{ occurs in } D \} \).

**Proof.** The proof defines a history-aware strategy \( S_D \) for \( \text{Reg}^⁺ \) as a modification of an arbitrary (history-free) strategy for \( \text{Reg}^⁺ \) lifted to a labelled version of \( \text{Reg}^⁺ \). Thereby the modification is performed according to a given derivation \( D \), and the construction will guarantee that (i), (ii), and (iii) hold. We establish the lemma only for the case of derivations in \( \text{Reg}^⁺ \), since the case of derivations in \( \text{Reg} \) can be treated analogously. So, let \( D \) be a derivation in \( \text{Reg}^⁺ \) with conclusion \((M)\).

In a first step we decorate \( D \) with position labels such that a derivation \( D^{(\text{hb})} \) with conclusion \( ε:(M) \) in the variant proof system \( \overline{\text{Reg}}^⁺ \) in Figure 14 is obtained. Note that the decoration process can be carried out in a bottom-up manner, where the label in the conclusion of a rule instance determines the label in the premise(s) if that is not an already labelled term, and where in the case of instances of the rule of (FIX) also the labels in marked assumptions are determined.

In a second step we use the decorated version \( D^{(\text{hb})} \) of \( D \) to define an history-aware strategy \( S_D \) according to which the term \((M)\) can be reduced as ‘prescribed’ by \( D^{(\text{hb})} \). Since the derivations can only determine the strategy \( S_D \) on terms occurring in \( D \), we also
have to define $\mathbb{S}_\mathcal{D}$ on other terms. This will be done by choosing an arbitrary (but here: history-free) scope-delimiting strategy $\mathbb{S}$ for $\mathcal{R}^*$, and basing the definition of $\mathbb{S}_\mathcal{D}$ on it.

We start by defining a labelling of $\mathcal{R}^*$ as the ARS for which $\mathbb{S}_\mathcal{D}$ will be defined as a history-free strategy, which together with an initial labelling $I$ then yields a history-aware strategy for $\mathcal{R}^*$. Assuming $\mathcal{R}^* = (\mathcal{Ter}(\lambda^\infty), \Phi, \mathcal{src}, \mathcal{tgt})$ as the formal representation of $\mathcal{R}^*$, we define the ARS $\mathcal{R}^\# = (\mathcal{Ter}(\lambda^\infty), \Phi, \mathcal{src}, \mathcal{tgt})$ where

\[
\begin{align*}
\overline{\mathcal{Ter}}(\lambda^\infty) & := \{ l : (\lambda\bar{y})N \mid (\lambda\bar{y})N \in \mathcal{Ter}(\lambda^\infty), l \in \{0,1\}^* \} \\
\overline{\Phi} & := \{ l : (\lambda\bar{y})N, \phi, l' : (\lambda\bar{y}')N' \mid (6.5) \text{ holds} \} \\
\end{align*}
\]

and where $\mathcal{src}, \mathcal{tgt} : \overline{\Phi} \to \overline{\mathcal{Ter}}(\lambda^\infty)$ are defined as projections on the first, and respectively, the third component of the triples that constitute steps in $\overline{\Phi}$. Then the relation

\[
\mathbb{L} := \{ (l : (\lambda\bar{y})N, l : (\lambda\bar{y})N) \mid (\lambda\bar{y})N \in \mathcal{Ter}(\lambda^\infty), l \in \{0,1\}^* \} \\
\cup \{ (\phi, (\lambda\bar{y})N, \phi, (\lambda\bar{y})N) \mid (\lambda\bar{y})N, \phi, (\lambda\bar{y})N) \in \overline{\Phi} \}
\]

is a labelling of $\mathcal{R}^\#$ to $\overline{\mathcal{R}^\#}$. As initial labelling we choose the function $I$ that is defined by $I : \mathcal{Ter}(\lambda^\infty) \to \overline{\mathcal{Ter}}(\lambda^\infty)$, $(\lambda\bar{x})M \mapsto e : (\lambda\bar{x})M$ and which adds the label ‘e’.

Now we define the strategy $\mathbb{S}_\mathcal{D} = (\overline{\mathcal{Ter}}(\lambda^\infty), \overline{\Phi}_{\mathcal{on-D}}^{(ib)}, \overline{\Phi}_{\mathcal{not-on-D}}^{(ib)}, \mathcal{src}', \mathcal{tgt}')$ with

\[
\overline{\Phi}_{\mathcal{on-D}}^{(ib)} := \{ l : (\lambda\bar{y})N, \phi, l' : (\lambda\bar{y}')N' \mid (6.6) \text{ holds} \} \\
\text{there is an instance of } (\lambda), (\Phi), \text{ or } (\mathbb{S}) \text{ in } \overline{\mathcal{R}^\#} \text{ with } l : (\lambda\bar{y})N \in \mathcal{Ter}(\lambda^\infty) \text{ in the conclusion and the term } l' : (\lambda\bar{y}')N' \text{ in the premise, and with} \phi : (\lambda\bar{y})N \to_{\mathcal{R}^*} (\lambda\bar{y}')N' \text{ (one of) the corresponding step(s) in } \overline{\mathcal{R}^*} \}
\]

\[
\overline{\Phi}_{\mathcal{not-on-D}}^{(ib)} := \{ l : (\lambda\bar{y})N, \phi, l' : (\lambda\bar{y}')N' \mid (6.7) \text{ holds} \} \\
l : (\lambda\bar{y})N \text{ does not occur in } \mathcal{D}^{(ib)}, \phi \text{ is a step according to } \mathbb{S}
\]

where $\mathcal{src}', \mathcal{tgt}'$ are the appropriate restrictions of $\mathcal{src}$ and $\mathcal{tgt}$.

Note that, by its definition, $\mathbb{S}_\mathcal{D}$ is a sub-ARS of $\overline{\mathcal{R}^\#}$. Now for showing that $\mathbb{S}_\mathcal{D}$ is a (history-aware) strategy for $\overline{\mathcal{R}^\#}$, it has to be established that $\mathbb{S}_\mathcal{D}$ is a history-free strategy for the lifted version $\overline{\mathcal{R}^\#}$ of $\mathcal{R}^*$. For this it remains to show that every normal
form of $S_D$ is also a normal form of \( \overline{\text{Reg}} \). So, let $l: (\lambda \bar{y}) N \in \text{Ter}((\Lambda^\infty))$ be such that it is not a normal form of \( \overline{\text{Reg}} \). Then also $\overline{(\lambda \bar{y}) N}$ is not a normal form of \( \overline{\text{Reg}} \). We will distinguish the cases that $l: (\lambda \bar{y}) N$ occurs on $D^{(lb)}$ or not for showing that there is a step in $S_D$ with this labelled term as a source.

For the first case, we suppose that $l: (\lambda \bar{y}) N$ does not occur in $D^{(lb)}$. Then there is a step $\phi: ((\lambda \bar{y}) N) \rightarrow ((\lambda \bar{y})' N')$ in the scope-delimiting strategy $S$ in $\overline{\text{Reg}}$, which gives rise to the step $\phi: ((\lambda \bar{y}) N) \rightarrow (l': (\lambda \bar{y})' N')$ in $\overline{\text{Reg}}$ and in $S_D$.

For the second case, we suppose that $l: (\lambda \bar{y}) N$ occurs in $D^{(lb)}$, and we fix an occurrence $o$. Since by assumption $l: (\lambda \bar{y}) N$ is not a normal form of $\overline{\text{Reg}}$, $o$ cannot be the occurrence of an axiom ($\emptyset$), and hence it is either an occurrence as the conclusion of an instance of one of the rules $\{(\lambda), (\emptyset), (S)\}$ in $D^{(lb)}$, or as a marked assumption in $D^{(lb)}$. If $o$ is the conclusion of an instance $\iota$ of $\{(\lambda), (\emptyset), (S)\}$, then $\iota$ defines a step on $l: (\lambda \bar{y}) N$ which also is a step in $S_D$. If $o$ is the conclusion of an instance of $\{(\text{FIX})\}$ in $D$, then we consider an arbitrary path $\pi$ in $D^{(lb)}$ from $o$ upwards towards a leaf of $D^{(lb)}$. Since, due to the side-condition of the rule (\text{FIX}), immediate subderivations of instances of (\text{FIX}) consist of at least one rule application, $\pi$ cannot consist merely of applications of (\text{FIX}). Hence by following $\pi$ from $o$ upwards, after a number of successive instances of (\text{FIX}), each of which have $l: (\lambda \bar{y}) N$ as conclusion and premise, an instance of one of the rules $\{(\lambda), (\emptyset), (S)\}$ follows, which witnesses a step with source $\{(\lambda), (\emptyset), (S)\}$ in $\overline{\text{Reg}}$ and in $S_D$. Finally, if $o$ is an occurrence in a marked assumption at the top of the prooftree $D^{(lb)}$, then, since $D^{(lb)}$ is a closed derivation and due to the form of instances of the assumption-discharging rule (\text{FIX}), there is also an occurrence $o'$ of $l: (\lambda \bar{y}) N$ as the conclusion of an instance of (\text{FIX}) in $D^{(lb)}$. Now the argument above can be applied to the occurrence $o'$ to obtain a step of $S_D$ on $l: (\lambda \bar{y}) N$.

By construction $S_D$ conforms to (i) and (ii) because of the inclusion of $\overline{\Phi}_{\not\in\text{on}}, D^{(lb)}$ and $\overline{\Phi}_{\not\in\text{on}}, D^{(lb)}$, respectively; (iii) follows from (ii).

Lemma 6.11 (from scope/scope\(^*\)-delim. strategies to Reg/Reg\(^*\)-derivations).
Let $M \in \text{Ter}(\Lambda^\infty)$, and let $S$ be a scope-delimiting strategy for $\overline{\text{Reg}}$ (a scope\(^*\)-delimiting strategy for $\overline{\text{Reg}}$) such that $\text{ST}_{\overline{\text{Reg}}}(M)$ is finite. Then there exists a closed derivation $D$ in $\overline{\text{Reg}}$ (in $\overline{\text{Reg}}\_0$, and hence in $\overline{\text{Reg}}$) with conclusion $\overline{\{M\}}$ such that the following properties hold (note the minor differences with the items (i), (ii), and (iii) in Lemma 6.10):

(i) Every (non-cyclic) path in $D$ from the conclusion upwards to a leaf of the prooftree $D$ corresponds to a $\rightarrow_{S}$-rewrite sequence starting on $\overline{\{M\}}$ that passes over instances of the rules $\{(\emptyset)\}$ to the left and to the right correspond to $\rightarrow_{a_0,S}$- and $\rightarrow_{a_1,S}$-steps, respectively, and passes over instances of $\{(\lambda)\}$ and of (Vacuous) $\{(S)\}$ correspond to $\rightarrow_{\lambda,S}$- and $\rightarrow_{\text{del},S}$-steps ($\rightarrow_{S}$-steps); passes from the conclusion to the premise of instances of (\text{FIX}) correspond to empty steps.

(ii) Every sufficiently long $\rightarrow_{S}$-rewrite sequence on $(\lambda \bar{x}) M$ has an initial segment that corresponds to a (non-cyclic) path in $D$ from the conclusion upwards to a leaf of the prooftree: thereby a $\rightarrow_{a_0,S}$- or $\rightarrow_{a_1,S}$-step corresponds to a pass over (possibly some (\text{FIX})-instances followed by) an instance of $\{(\emptyset)\}$ in direction left and right, respectively; a $\rightarrow_{\lambda,S}$- or $\rightarrow_{\text{del},S}$-step ($\rightarrow_{S}$-step) corresponds to a pass over (possibly some (\text{FIX})-instances followed by) an instance of $\{(\lambda)\}$ or (Vacuous) $\{(S)\}$, respectively.
(iii) $ST_\gamma(M) \subseteq \{(\lambda \bar{y})N \mid \text{the term } (\lambda \bar{y})N \text{ occurs in } D\}.$

**Proof.** We will argue only for the part of the statement of the lemma concerning a scope"-delimiting strategy for $\text{Reg}^\ast$, since the case with a scope-delimiting strategy for $\text{Reg}$ can be established analogously.

Let $M$ be an infinite $\lambda$-term, and let $S$ be a scope"-delimiting strategy for $\text{Reg}$ such that $ST_\gamma(M)$ is finite. Now let $D_0$ be the (trivial) derivation with conclusion $(\lambda y)M$, which, in case that this is not an axiom of $\text{Reg}_0^\ast$ (and $\text{Reg}^\ast$), is also an assumption, and then is of the form $(\lambda y)N$, carrying an assumption marker $u$. If $D_0$ is an axiom, then it is easy to verify that the statements (i), (ii), and (iii) hold.

Otherwise we construct a sequence $D_1, D_2, \ldots$ of derivations where each $D_n$ satisfies the properties (i), (ii), and (iii), terms in marked assumptions are not also terms in axioms $\emptyset$, and where $D_{n+1}$ extends $D_n$ by one additional rule instance above a marked assumption in $D_n$. For the extension step on a derivation $D_n$, a marked assumption $((\lambda y)N)^u$ in $D_n$ is picked with the property that the term $((\lambda y)N)$ does not appear in the thread down to the conclusion of $D_n$.

Suppose that the $\to_S$-rewrite sequence from the conclusion of $D_n$ up to the marked assumption is of the form:

$$\tau : (\lambda y)M_0 \to_S (\lambda y)M_1 \to_S \ldots \to_S (\lambda y)M_m = (\lambda y)N$$

Note that, since by assumption $(\lambda y)N$ is not a term in an axiom $0$ of $\text{Reg}_0^\ast$, it follows by Proposition 4.8, (vi), that it is not a $\to_{\text{reg}}$-normal form. Then depending on whether the possible next step(s) in an $\to_S$-rewrite that extends $\tau$ by one step is a $\to_{\lambda, S}$, $\to_{\text{del}, S}$-step, or either a $\to_{\emptyset, S}$- or a $\to_{\emptyset, \emptyset}$-steps, the derivation $D_n$ is extended above the marked assumption $((\lambda y)N)^u$ by an application of $\lambda, S$, or $\emptyset$, respectively. For example in the case that $\tau$ extends by one additional step to either of the two rewrite sequences:

$$\tau_i : (\lambda y)M = (\lambda \bar{x}_0)M_0 \to_S \ldots \to_S (\lambda \bar{x}_m)M_m = (\lambda \bar{x}_m)M_{m,0} \to_{\emptyset, S} (\lambda \bar{x}_m)M_{m,1}$$
with $i \in \{0, 1\}$, the derivation $D_n$ of the form:

$$\begin{align*}
&\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle^u \\
D_n &\quad \text{is extended to } D_{n+1} : \\
\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle^u &\quad \text{by} \\
D_n &\quad D_n \quad \text{by} \\
\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle^u &\quad \text{at} \\
\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle &\quad \text{by} \\
D_n &\quad D_n \quad \text{by} \\
\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle^u &\quad \text{at} \\
\langle (\lambda \bar{x}_m)M_{m,0}M_{m,1} \rangle &\quad \text{by} \\
D_n &\quad D_n \quad \text{by}
\end{align*}$$

for two fresh assumption markers $u_0$ and $u_1$ (the angle brackets $\langle \ldots \rangle$ are used here to indicate just a single formula occurrence at the top of the prooftree $D_n$). If either of $(\lambda \bar{x}_m)M_{m,0}$ or $(\lambda \bar{x}_m)M_{m,1}$ is an axiom, then the assumption marker is removed and the formula is marked as an axiom $0$, accordingly. Note that, if the statements (i), (ii), and (iii) are satisfied for $D = D_n$, then this is also the case for $D = D_{n+1}$. Furthermore, terms in marked assumptions are not terms in axioms of $\text{Reg}_0^\ast$.

The extension process continues as long as $D_n$ contains a marked assumption $((\lambda y)N)^u$ without a $\text{Reg}_0^\ast$-admissible repetition" beneath it, by which we mean the occurrence $o$ of the term $(\lambda y)N$ on the thread down to the conclusion in $D$, but strictly beneath the marked assumption, such that furthermore all terms on the part of the thread down to $o$
have an abstraction prefix of length greater or equal to $|y|$. (Note the connection to the side-condition on instances of the rule (FIX) in $\text{Reg}_{0}^{*}$, and, in particular, that marked assumptions with an $\text{Reg}_{0}^{*}$-admissible repetition beneath it could be discharged by an appropriately introduced instance of (FIX) in $\text{Reg}_{0}^{*}$.)

That the extension process terminates can be seen as follows: Suppose that, to the contrary, it continues indefinitely. Then, since the derivation size increases strictly in every step, an infinite proof tree $\mathcal{D}^{\infty}$ is obtained in the limit, which due to finite branchingness of the proof tree and König’s Lemma possesses an infinite path $\pi$ starting at the conclusion. Now note that due to (i), $\pi$ corresponds to an infinite $\Rightarrow_{\Sigma}$-rewrite sequence. Due to Proposition 4.29, this infinite rewrite sequence must contain a grounded cycle. However, the existence of such grounded cycle contradicts the termination condition of the extension process, because every grounded cycle provides an $\text{Reg}_{0}^{*}$-admissible repetition.

Let $\mathcal{D}_{N}$, for some $N \in \mathbb{N}$, be the derivation that is reached when no further extension step, as described, is possible. By the construction the statements (i), (ii), and (iii) are satisfied for $\mathcal{D} = \mathcal{D}_{N}$. Furthermore, $\mathcal{D}_{N}$ is a derivation in $\text{Reg}^{*}$ and $\text{Reg}_{0}^{*}$ in which every leaf at the top is either an axiom 0 or an assumption $((\lambda y)N)^{u}$ marked with a unique marker $u$, and for every such marked assumption in $\mathcal{D}_{N}$, there is a $\text{Reg}_{0}^{*}$-admissible repetition strictly beneath it. This fact enables us to modify $\mathcal{D}_{N}$ into a closed derivation in $\text{Reg}_{0}^{*}$ by closing all open assumptions by newly introduced applications of the rule (FIX). More precisely, steps of the following kind are carried out repeatedly. A derivation with $\mathcal{D}$ repetition strictly beneath it. This fact enables us to modify $\mathcal{D}_{N}$ into a closed derivation in $\text{Reg}_{0}^{*}$ by closing all open assumptions by newly introduced applications of the rule (FIX). More precisely, steps of the following kind are carried out repeatedly. A derivation with occurrence of a number of marked assumptions $((\lambda y)N)^{u}$ highlighted together with a single occurrence of the term $(\lambda y)N$ in its interior that indicates the $\text{Reg}_{0}^{*}$-admissible repetition for the displayed marked assumptions:

$$
\frac{\{(\lambda y)N^{u_{1}}\} \ldots \{(\lambda y)N^{u_{k}}\}}{\mathcal{D}_{000}}
\frac{\{(\lambda y)N\}}{\mathcal{D}_{00}}
\frac{()M}{(\lambda y)N} \text{ is modified into:}
\frac{\{(\lambda y)N\} \text{ FIX, } w}{(\lambda y)N}
$$

where $w$ is a fresh assumption marker. In every such transformation step the number of open assumptions is strictly decreased, but the properties (i), (ii), and (iii) (for $\mathcal{D}$ the resulting derivation of such a step) is preserved. Hence after finitely many such transformation steps a derivation $\mathcal{D}$ in $\text{Reg}_{0}^{*}$ without open assumptions and with the properties (i), (ii), and (iii) is reached, and obtained as the result of this construction.

As a consequence of the two lemmas above, derivability in $\text{Reg}^{*}$ and in $\text{Reg}_{0}^{*}$ coincides.

**Proposition 6.12.** For all infinite $\lambda$-terms $M$: $\vdash_{\text{Reg}^{*}} ()M$ if and only if $\vdash_{\text{Reg}_{0}^{*}} ()M$.

**Proof.** The direction “$\Rightarrow$” follows by the fact that every derivation in $\text{Reg}_{0}^{*}$ is also a derivation in $\text{Reg}^{*}$. For the direction “$\Leftarrow$”, let $M$ be an infinite term such that $\vdash_{\text{Reg}^{*}} ()M$. By Lemma 6.10 there exists a scope $\Rightarrow$-delimiting strategy $\mathcal{S}$ such that $\text{ST}_{\Rightarrow}(M)$ is finite. But then it follows by Lemma 6.11 that there is also a closed derivation in $\text{Reg}_{0}^{*}$ with conclusion $()M$, and hence that $\vdash_{\text{Reg}_{0}^{*}} ()M$. \qed
Now we have assembled all auxiliary statements that we use for proving a theorem that tightly links derivability in the proof system \( \text{Reg} \) with regularity, and derivability in \( \text{Reg}^* \) and in \( \text{Reg}^*_0 \) with strong regularity, of infinite \( \lambda \)-terms.

**Theorem 6.13.** The following statements hold for the proof systems \( \text{EQ}^\infty \) and \( \text{EQ}^\infty \).

(i) \( \text{Reg} \) is sound and complete for regular infinite \( \lambda \)-terms. That is, for all \( M \in \text{Ter}(\lambda^\infty) \):

\[
\vdash_{\text{Reg}} (\cdot) M \quad \text{if and only if} \quad M \text{ is regular.}
\]

(ii) \( \text{Reg}^* \) and \( \text{Reg}^*_0 \) are sound and complete for strongly regular infinite \( \lambda \)-terms. That is, for all \( M \in \text{Ter}(\lambda^\infty) \) the following assertions are equivalent:

(a) \( M \) is strongly regular.

(b) \( \vdash_{\text{Reg}^*} (\cdot) M \).

(c) \( \vdash_{\text{Reg}^*_0} (\cdot) M \).

**Proof.** Since the proof of statement (ii) of the theorem can be carried out analogously (taking into account Proposition 6.12), we argue here only for statement (i).

For "\( \Rightarrow \)" in (i), let \( M \) be an infinite \( \lambda \)-term that is regular. Then there exists a scope-delimiting strategy \( S \) on \( \text{Reg} \) such that \( ST_S(M) \) is finite. By Lemma 6.11 it follows that there exists a closed derivation \( D \) in \( \text{Reg} \) with conclusion \( (\cdot) M \). This derivation witnesses \( \vdash_{\text{Reg}} (\cdot) M \). For "\( \Leftarrow \)" in (i), suppose that \( \vdash_{\text{Reg}} (\cdot) M \). Then there exists a closed derivation \( D \) in \( \text{Reg} \) with conclusion \( (\cdot) M \). Now Lemma 6.10 entails the existence of a scope-delimiting strategy \( S \) in \( \text{Reg}^* \) such that, in particular, \( ST_S(M) \) is finite. This fact implies that \( M \) is regular.

For defining, shortly, of a proof system for equality of strongly regular infinite \( \alpha \)-terms, we first give a specialised version of the proof system \( \text{EA}^\alpha \) for \( \alpha \)-equivalence of iCRS-preterms for preterms, and a corresponding system on terms, in \( (\lambda^\infty) \).

**Definition 6.14 (proof systems \( \text{EQ}^\alpha \), \( \text{EQ}^\infty \)).** The proof system \( \text{EQ}^\alpha \) for \( \alpha \)-equivalence of infinite preterms in \( (\lambda^\infty) \) consists of the rules displayed in Figure 15. The proof system \( \text{EQ}^\infty \) for equality of infinite terms in \( (\lambda^\infty) \) consists of the rules displayed in Figure 16. Provability in \( \text{EQ}^\alpha \) and in \( \text{EQ}^\infty \) is defined, analogously to the proof system
\[ \frac{(\lambda z)y = (\lambda z)u}{(\lambda z)y} \cdot 0 \quad \frac{(\lambda z)M = (\lambda z)N}{(\lambda z)M} = (\lambda z)N \quad S \quad (\text{if } y \text{ does not occur in } M, \text{ and } w \text{ does not occur in } N) \]

\[ \frac{(\lambda z)M = (\lambda z)N}{(\lambda z)\lambda y.M} = (\lambda z)\lambda u.N} \quad \lambda \quad (\lambda z)M_0 = (\lambda \bar{y})N_0 \quad (\lambda \bar{z})M_1 = (\lambda \bar{y})N_1 \quad \emptyset \]

Fig. 16. Proof system \(\text{EQ}^\infty\) for equality of terms in \((\lambda^\infty)\) in informal notation.

\(\mathcal{A}^\infty\) from Definition 2.3, as the existence of a completed (possibly infinite) derivation, and will, as was first done so for \(\mathcal{A}^\infty\) in Definition 2.2, be indicated using the symbol \(\approx_{\infty}\).

**Proposition 6.15.** The following statements hold for the proof systems \(\text{EQ}^\infty\) and \(\text{EQ}^\infty\).

(i) \(\text{EQ}_\alpha^\infty\) is sound and complete for \(\approx_\alpha\) on \((\lambda^\infty)\)-preterms. That is:

\[ \approx_{\infty}\text{EQ}^\infty \quad \text{pre}_\alpha(s) = \text{pre}_\alpha(t) \quad \text{if and only if} \quad s =_\alpha t. \]

holds for all closed preterms \(s\) and \(t\) in \((\lambda^\infty)\).

(ii) \(\text{EQ}^\infty\) is sound and complete for equality between \((\lambda^\infty)\)-terms. That is:

\[ \approx_{\infty}\text{EQ}^\infty \quad M = N \quad \text{if and only if} \quad M = N. \]

Proof. For statement (i) it suffices to show that, for an equation \(\text{pre}_\alpha(s) = \text{pre}_\alpha(t)\) between preterms of \((\lambda^\infty)\), derivability in \(\text{EQ}^\infty_\alpha\) coincides with derivability of this equation in the general proof system \(\mathcal{A}^\infty\) for \(\alpha\)-equivalence between iCRS-preterms in Definition 2.3. Given a derivation \(D^\infty\) in \(\text{EQ}^\infty_\alpha\), a derivation \(D^1\) in \(\mathcal{A}^\infty\) results by replacing each formula occurrence \(\text{pre}_\alpha([x_1 \ldots x_n]s) = \text{pre}_\alpha([y_1 \ldots y_n]t)\) by the formula occurrence \([x_1 \ldots x_n]s = [y_1 \ldots y_n]t\) and adding an instance of the rule for the function symbol \(\text{pre}_\alpha\) at the bottom. Then instances of the axioms and rules \((0), \quad (\overline{0}), \quad (\lambda), \quad (\text{and })\) and \((\text{S})\) in \(D^\infty\) correspond to instances of axioms and rules \((0), \quad (\text{app}), \quad ([]), \quad \text{and } (\text{S})\) in \(D^1\), respectively. This proof transformation also has an inverse.

For \(\approx_{\text{S}}\) in statement (ii) it suffices to note: Every scope\(^*\)-delimiting strategy on \(\text{Reg}\) can be used to stepwisely extend finite derivations in \(\text{EQ}^\infty\) with conclusion \(\{\} M = \{\} M\) by one additional rule application above a leaf containing a formula \((\lambda y)N = (\lambda y)N\) that is not an axiom \(0\), which implies that \((\lambda y)N\) is not a normal form of \(\rightarrow_{\text{reg}}\). If these extensions are carried out in a fair manner by extending all non-axiom leaves at depth \(n\) in the prooftree before proceeding with leaves at depth \(> n\), then in the limit a completed derivation in \(\text{EQ}^\infty\) is obtained.

For \(\Rightarrow_{\text{S}}\) in statement (ii), suppose that \(D^\infty\) is a completed derivation in \(\text{EQ}^\infty\) with conclusion \(\{\} M = \{\} N\). Let \(\text{pre}_\alpha(s)\) and \(\text{pre}_\alpha(t)\) be preterm representatives of \(\{\} M\) and \(\{\} N\), respectively. Now a completed derivation \(D^\infty_{\text{pter}}\) in \(\text{EQ}^\infty_\alpha\) can be found by developing it stepwisely from the conclusion \(\text{pre}_\alpha(s) = \text{pre}_\alpha(t)\) upwards, parallel to \(D^\infty\), and following the rules of \(\text{EQ}^\infty_\alpha\), which are invertible (that is, the premise/s of a rule instance is/are uniquely determined by the conclusion). Then \(D^\infty_{\text{pter}}\) is a preterm representative version
expressibility in the lambda calculus with letrec

\[(\lambda x)M = (\lambda y)N\]

\[D_0\]

\[(\lambda x)M = (\lambda y)N\]

\[(\lambda x)M = (\lambda y)N\] FIX, u (if \(|D_0| \geq 1\))

Fig. 17. The rule (\text{FIX}), which is added to the rules of \text{EQ}^\infty from Figure 16 in order to obtain the proof system \text{Reg}^*_\text{letrec} for equality of strongly regular infinite \(\lambda\)-terms.

of \(\mathcal{D}^\infty\). By using (i), it follows that \(s \equiv_\alpha t\). Since \(s\) and \(t\) are preterm representatives of \(M\) and \(N\), respectively, \(M = N\) follows.

\textbf{Definition 6.16 (the proof system Reg}^{*}\text{).} The natural-deduction-style proof system \text{Reg}^*_\text{letrec} for equality of strongly regular, infinite \(\lambda\)-terms has all the rules of the proof system \text{EQ}^\infty from Definition 6.14 and Figure 16, and additionally, the rule (\text{FIX}) in Figure 17. But contrary to the definition in \text{EQ}^\infty, provability of an equation \((\lambda x)M = (\lambda y)N\) in \text{Reg}^*_\text{letrec} is defined as the existence of a finite closed derivation with conclusion \((\lambda x)M = (\lambda y)N\).

\textbf{Theorem 6.17.} \text{Reg}^*_\text{letrec} is sound and complete for equality between strongly regular, infinite \(\lambda\)-terms. That is, for all strongly regular, infinite \(\lambda\)-terms \(M\) and \(N\) it holds:

\[
\vdash_{\text{Reg}^*_\text{letrec}} M = N \quad \text{if and only if} \quad M = N.
\]

\textit{Sketch of the Proof.} Let \(M\) and \(N\) be strongly regular, infinite \(\lambda\)-terms. In view of Proposition 6.15, (ii), it suffices to show:

\[
\vdash_{\text{Reg}^*_\text{letrec}} M = N \quad \text{if and only if} \quad \vdash_{\text{EQ}^\infty} M = N.
\] (6.8)

For showing “\(\Leftarrow\)” in (6.8), let \(\mathcal{D}^\infty\) be a derivation in \text{EQ}^\infty with conclusion \((\lambda x)M = (\lambda y)N\). Since paths in \(\mathcal{D}^\infty\) correspond to \(\rightarrow_{reg}\)-rewrite sequences, and since the number of generated subterms of both \(M\) and \(N\) are finite (as a consequence of their strong regularity), on every infinite thread equation repetitions occur. These repetitions can be used to cut all infinite threads by appropriate introductions of instances of (\text{FIX}) in order to obtain a finite and closed derivation in \text{Reg}^*_\text{letrec} with the same conclusion.

For showing “\(\Rightarrow\)” in (6.8), let \(\mathcal{D}\) be a closed derivation in \text{Reg}^*_\text{letrec} with conclusion \((\lambda x)M = (\lambda y)N\). Now \(\mathcal{D}\) can be unfolded into an infinite derivation \(\mathcal{D}^\infty\) in \text{EQ}^\infty by repeatedly removing a bottommost instance of (\text{FIX}) and inserting its immediate subderivation above each of the marked assumptions the instance discharges. If this process is organised in a fair manner with respect to bottommost instances of (\text{FIX}), then in the limit an infinite completed prooftree with conclusion \((\lambda x)M = (\lambda y)N\) in \text{EQ}^\infty is obtained. For productivity of this process it is decisive that the side-condition on every instance \(\iota\) of the rule (\text{FIX}) guarantees that on threads from the conclusion of \(\iota\) to a marked assumption that is discharged by \(\iota\) at least one instance of a rule different from (\text{FIX}) is passed.

For the purpose of the following definition and the respective proof system in Figure 18 we extend the signature \(\Sigma_{\lambda}\text{letrec}\) of \text{Reg}_{\lambda}\text{letrec} and \text{Reg}^*_\text{letrec} by an infinite set of constants for which we use the symbol \(c\) as syntactical variables which frequently carry index subscripts.
\[
\begin{align*}
\{ \{ (\lambda \vec{x}) \varepsilon_{j_i} \} \}_{i=1, \ldots, n} & \quad \{ \{ (\lambda \vec{x}) \varepsilon_{j_i} \} \}_{i=1, \ldots, n} \\
\{ \ldots \} (\lambda \vec{x}) \; \text{letrec} \; f_1 = L_1 \ldots f_n = L_n \; \text{in} \; L_{n+1} & \quad (\lambda \vec{x}) \; \text{letrec} \; f_1 = L_1 \ldots f_n = L_n \; \text{in} \; L_{n+1} \\
(\lambda \vec{x}) \; \text{letrec} \; f_1 = L_1 \ldots f_n = L_n \; \text{in} \; L_{n+1} & \quad \text{FIX}_{\text{letrec}}, u_1, \ldots, u_n \\
\end{align*}
\]

where \( c_{f_1}, \ldots, c_{f_n} \) are distinct constants fresh for \( L_1, \ldots, L_{n+1}, \) and substitutions \( L_i[f := \varepsilon_{j_i}] \) stands short for \( L_i[f_1 := c_{f_1}, \ldots, f_n := c_{f_n}] \).

**Definition 6.18 (proof systems Reg\textsubscript{letrec}, Reg\textsuperscript{*}\textsubscript{letrec}, and ug-Reg\textsubscript{letrec}, ug-Reg\textsuperscript{*}\textsubscript{letrec}).**

The proof systems \( \text{Reg}^*\textsubscript{letrec} \) and \( \text{Reg}\textsubscript{letrec} \) for \( \lambda_{\text{letrec}} \)-terms arise from the proof systems \( \text{Reg} \) and \( \text{Reg}^* \) (see Definition 6.3, Figure 12 and Figure 13), respectively, by replacing the terms in the axioms (\( \emptyset \)), and rules (\( \lambda \)), (\( \emptyset \)), (S) and (Vacuous) through \( \lambda_{\text{letrec}} \)-terms with abstraction prefixes accordingly, and by replacing the rule (\( \text{FIX} \)) with the rule (\( \text{FIX}_{\text{letrec}} \)) in Figure 18. The side-condition concerning access path cycles on the derivation arising by an instance of (\( \text{FIX}_{\text{letrec}} \)) pertains only to bottommost occurrences of this rule, and is explained below. By (\( \text{FIX}_{\text{letrec}} \)) we mean the variant of the rule (\( \text{FIX}_{\text{letrec}} \)) in which the side-condition concerning guardedness of the arising derivation on access path cycles has been dropped. By \( \text{ug-Reg}_{\text{letrec}}, \text{ug-Reg}^*_{\text{letrec}} \) we denote the variants of \( \text{Reg}_{\text{letrec}}, \text{Reg}^*_{\text{letrec}} \), respectively, in which the rule (\( \text{FIX}_{\text{letrec}} \)) is replaced by the rule (\( \text{FIX}_{\text{letrec}} \)).

Let \( \mathcal{D} \) be a derivation in one of these proof systems. By an **access path** of \( \mathcal{D} \) we mean a (possibly cyclic) path \( \pi \) in \( \mathcal{D} \) such that:

(a) \( \pi \) starts at the conclusion and can proceed in upwards direction;
(b) at instances of (\( \emptyset \)), \( \pi \) can step from the conclusion to one of the premises;
(c) at instances of (\( \text{FIX}_{\text{letrec}} \)), \( \pi \) can step from the conclusion to the rightmost premise (which corresponds to the in-part of the \( \text{letrec} \)-term that is parsed by this instance);
(d) when arriving at a marked assumption (\( (\lambda \vec{x}) \varepsilon_{j_i} \)\(^{\text{uni}} \)) that is discharged at an application of (\( \text{FIX}_{\text{letrec}} \)) of the form as displayed in Figure 18, \( \pi \) can step over to the conclusion (\( \lambda \vec{x} \) \( L_i[f := \varepsilon_{j_i}] \)) of the subderivation \( \mathcal{D}_i \) of that application of (\( \text{FIX}_{\text{letrec}} \)), and proceed from there, again in upwards direction.

For every formula occurrence \( o \) in \( \mathcal{D} \), by a **relative access path** from \( o \) we mean a path with the properties (b)-(d) that starts at \( o \) and proceeds in upwards direction. An access path (or relative access path) in \( \mathcal{D} \) is **cyclic** if there is a formula occurrence in \( \mathcal{D} \) that is visited more than once.

Now we say that \( \mathcal{D} \) is **guarded on access path cycles** if every cyclic access path contains, on each of its cycles, at least one guard, that is, an instance of a rule (\( \lambda \)) or (\( \emptyset \)). We say that \( \mathcal{D} \) is **guarded** if every relative access path contains a guard on each of its cycles.
Example 6.19. The λ_{letrec}-term \( L_1 = \text{letrec} \ f, g = \lambda x. x \ \text{in} \ \lambda y. g \) admits the following closed derivation \( D_1 \) in \( \text{Reg}_{letrec}/\text{Reg}_{letrec}^* \):

\[
\begin{array}{ccc}
(\lambda x. x) & 0 & (\lambda y) c_g \\
(\lambda y) c_f & \lambda y & (\lambda y) c_f c_g \\
(\lambda y) c_g & \lambda & (\lambda y) c_g c_f \\
\text{Vacuous}/\text{S} & \text{Vacuous}/\text{S} & \text{FIX}_{letrec}, u_1, u_2 \\
\text{letrec} f = f, g = \lambda x. x \ \text{in} \ \lambda y, g \\
\end{array}
\]

This derivation can be built in a straightforward way, from the bottom upwards. Note that \( D \) is guarded on access path cycles, and hence that the instance of (FIX_{letrec}) at the bottom is a valid one, because: \( D \) does not possess any cyclic access paths. In particular, there is no access path in \( D_1 \) that reaches the first premise of the instance of (FIX_{letrec}): this premise is the starting point of an unguarded relative access path, which entails that \( D_1 \) itself is not guarded.

Now consider the λ_{letrec}-term \( L_2 = \text{letrec} \ f, g = \lambda x. x \ \text{in} \ \lambda y, f \ g \). When trying to construct a derivation in \( \text{Reg}_{letrec}/\text{Reg}_{letrec}^+ \) for this term in a bottom-up manner, one arrives at the closed derivation \( D_2 \) in \( \text{ug-Reg}_{letrec}/\text{ug-Reg}_{letrec}^+ \):

\[
\begin{array}{ccc}
(\lambda x. x) & 0 & (\lambda y) c_g \\
(\lambda y) c_f & \lambda y & (\lambda y) c_f c_g \\
\text{Vacuous}/\text{S} & \text{Vacuous}/\text{S} & (\lambda y) c_g c_f \\
\text{FIX}_{letrec}, u_1, u_2 & \text{but not (FIX_{letrec})} \\
\text{letrec} f = f, g = \lambda x. x \ \text{in} \ \lambda y, f \ g \\
\end{array}
\]

However, \( D_2 \) is not a valid derivation in \( \text{Reg}_{letrec}/\text{Reg}_{letrec}^+ \), as the inference step at the bottom is an instance of (FIX_{letrec}), but not of (FIX_{letrec}), because the side-condition on the arising derivation to be guarded on access path cycles is not satisfied: now there is an access path that reaches the first premise of the derivation and that continues looping on this an unguarded cycle. Since the bottom-up search procedure for derivations is deterministic in this case, it follows that \( (\lambda) L_2 \) is not derivable in \( \text{Reg} \) nor in \( \text{Reg}^+ \).

Similar as the correspondence, stated by Proposition 6.9, between (possibly cyclic) paths in a derivation in \( \text{Reg} \) and \( \text{Reg}^+ \) starting at the conclusion and rewrite sequences with respect to \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^+} \) on the infinite term in the conclusion, there is also the following correspondence between access paths in a derivation in \( \text{Reg}_{letrec} \) and \( \text{Reg}_{letrec}^+ \), and rewrite sequences with respect to \( \rightarrow_{\text{reg}} \) and \( \rightarrow_{\text{reg}^+} \) on the λ_{letrec}-term in the conclusion.

Proposition 6.20. Let \( D \) be a derivation in \( \text{Reg}_{letrec} \) or \( \text{ug-Reg}_{letrec} \) (in \( \text{Reg}_{letrec}^+ \) or in \( \text{ug-Reg}_{letrec}^+ \)) with conclusion \( (\lambda \bar{x}) L \).

(i) Then every access path in \( D \) to an occurrence \( o \) of a term \( (\lambda \bar{y}) P \) corresponds to a \( \rightarrow_{\text{reg}} \)-rewrite sequence \( (\lambda \bar{x}) L \rightarrow_{\text{reg}} (\lambda \bar{y}) \text{letrec} \ B \) in \( \hat{P} \) (to a \( \rightarrow_{\text{reg}^+} \)-rewrite sequence \( (\lambda \bar{x}) L \rightarrow_{\text{reg}^+} (\lambda \bar{y}) \text{letrec} \ B \) in \( \hat{P} \)), where \( B \) arises as the union of all outermost binding groups in conclusions of instances of (FIX_{letrec}) below \( o \), and \( (\lambda \bar{y}) P = (\lambda \bar{y}) \hat{P}[\hat{f} := \bar{c} f] \) where \( \hat{f} \) is comprised of the recursion variables occurring in \( B \) and \( \bar{c} f \) distinct constants for \( \hat{f} \) as chosen by \( D \). More precisely:
(a) a pass over an instance of (\text{FIX}_{\text{letrec}}) corresponds to an empty or $\rightarrow_{\varphi, \text{letrec}}$-step, dependent on whether the instance is the bottommost (\text{FIX}_{\text{letrec}})-instance or not;
(b) a pass over an instance of the rule (\text{\@}) to the left/to the right corresponds to a $\rightarrow_{\alpha_n}$-step/$\rightarrow_{\alpha_1}$-step, which, if the application is somewhere above an instance of (\text{FIX}_{\text{letrec}}), has to be preceded by a $\rightarrow_{\varphi, \alpha}$-step;
(c) a pass over an instance of the rule (\text{\_}) corresponds to a $\rightarrow_{\lambda}$-step which, if the application is above an instance of (\text{FIX}_{\text{letrec}}), has to be preceded by a $\rightarrow_{\varphi, \lambda}$-step;
(d) a pass over an instance of the rule Vacuous (\text{\$}) corresponds to a $\rightarrow_{\text{del}}$-step ($\rightarrow_{\text{s}}$-step), possibly preceded by an application of $\rightarrow_{\varphi, \text{red}}$;
(e) a step from a marked assumption to a premise of a (\text{FIX}_{\text{letrec}})-instance, a step as described in item (c) of the definition of access paths, corresponds to an $\rightarrow_{\varphi, \text{rec}}$-step followed by a $\rightarrow_{\varphi, \text{red}}$-step.

(ii) $L$ is reduced if every formula occurrence in $D$ can be reached by an access path.

**Example 6.21.** The rewrite cycle in Example 3.19 that witnesses that the $\lambda_{\text{letrec}}$-term considered there, $\text{letrec} f = \text{letrec} g = f \text{ in } g \text{ in } f$, is not unfoldable can also be recognised, using the statement of Proposition 6.20, from the following derivation in $\text{ug-Reg}_{\text{letrec}}^*$:

$$
\frac{\gamma}{\text{letrec} f = \text{letrec} g = f \text{ in } g \text{ in } f} \frac{\gamma}{\text{FIX}_{\text{letrec}}, \text{ Fix} f} \frac{\gamma}{\text{FIX}_{\text{letrec}}}, \text{ Fix} f
$$

Note that the instance of (\text{FIX}_{\text{letrec}}) at the bottom is not an instance of (\text{FIX}_{\text{letrec}}), since it is not guarded (has an unguarded cyclic access path that reaches and cycles on the left premise of the instance of (\text{FIX}_{\text{letrec}})).

**Lemma 6.22.** Let $D$ be a closed derivation in $\text{ug-Reg}_{\text{letrec}}$ (in $\text{ug-Reg}_{\text{letrec}}^*$) with conclusion (\text{\_}). Then there exists a scope-delimiting (scope$^\ast$-delimiting) strategy $S_D$ for $\text{Reg}_{\text{letrec}}$ (for $\text{Reg}_{\text{letrec}}^\ast$) with the following properties:

(i) Every access path in $D$ corresponds to a rewrite sequence with respect to $S_D$ starting on (\text{\_}) in the sense of Proposition 6.20.
(ii) Every rewrite sequence that starts on (\text{\_}) and proceeds according to $S_D$ corresponds to an access path in $D$ with correspondences as described in Proposition 6.20, (i).
(iii) $ST_{S_D}(L) = \{ (\lambda \beta) \text{letrec} B \text{ in } \bar{P} \mid (\lambda \beta) \text{letrec} B \text{ in } \bar{P} \text{ arises from an occ. of } (\lambda \beta) P \text{ on an access path of } D \text{ as described in Prop. 6.20, (i)} \}$.

As a consequence of that $D$ is finite, $L$ is $S_D$-regular.
(iv) $L$ is $S_D$-productive $\iff$ $D$ is guarded (i.e. $D$ derivation in $\text{Reg}_{\text{letrec}}^\ast$ ($\text{Reg}_{\text{letrec}}^\ast$).

**Proof.** Given a closed derivation $D$ with conclusion (\text{\_}) $L$ (for example) in $\text{ug-Reg}_{\text{letrec}}^\ast$, a scope$^\ast$-delimiting strategy $S_D$ for $\text{Reg}_{\text{letrec}}^\ast$ such that (i)–(iv) hold can be extracted from $D$ similar as in the proof of a scope$^\ast$-delimiting strategy $S_D$ in $\text{Reg}^\ast$ was extracted from a closed derivation in $\text{Reg}^\ast$. That the extracted strategy $S_D$ is productive/not productive for $L$ if $D$ is guarded/not guarded can be seen by the fact that $S_D$-rewrite sequences correspond to access paths of $D$ in the sense as stated by Proposition 6.20.

Now we will prove that derivability in $\text{ug-Reg}_{\text{letrec}}^\ast$/$\text{ug-Reg}_{\text{letrec}}^\ast$ is guaranteed for all
\[ \lambda_{letrec}\text{-terms}, \text{and that derivability in } \text{Reg}_{letrec}^\ast / \text{Reg}_{letrec} \text{ is a property of a } \lambda_{letrec}\text{-term that is decidable by an easy parsing process.} \]

**Proposition 6.23.** The following statements hold:

(i) For every \( \lambda_{letrec}\)-term \( L \), \( ()L \) is derivable both in \( ug-\text{Reg}_{letrec} \) and in \( ug-\text{Reg}_{letrec}^\ast \).

(ii) For every \( \lambda_{letrec}\)-term \( L \), derivability of \( ()L \) in \( Reg_{letrec}^\ast \) is decidable in at most quadratic time in the size of \( L \).

**Proof.** For (i) note that for every \( \lambda_{letrec}\)-term \( L \), a closed derivation \( D_L \) with conclusion \( ()L \) in \( ug-\text{Reg}_{letrec}^\ast \) can be produced by a bottom-up construction following the term structure of \( L \). Hereby use of the rules \((S)\) can be restricted to instances immediately below marked assumptions such that, viewed from a (non-cyclic) path \( \pi \) from the conclusion upwards to a marked assumption, these \((S)\)-instances are only introduced to shorten the frozen abstraction prefixes by all \( \lambda \)-abstractions that have become frozen on \( \pi \) (in order to conform to the side-condition on \((\text{FIX}_{letrec})\)-instances to have the same frozen abstraction prefix lengths in the discharged marked assumptions as in the conclusion and in the premises).

Now for (ii) in order to decide derivability of \( ()L \) in \( \text{Reg}_{letrec}^\ast \), it suffices to decide whether the derivation \( D_L \) in \( ug-\text{Reg}_{letrec}^\ast \) obtained as described above, or its bottommost instance of \((\text{FIX}_{letrec})\) if there is any, is guarded on all of its access path cycles. (Note that in the construction of \( D_L \) only the freedom in placing instances of \((S)\) has been used in a certain, namely lazy, way. The specific placement of instances of these rules does not interfere with the existence or non-existence of guards, that is instances of \( \lambda \) or @ on cycles of access paths.) For this it remains to check whether every cycle on an access path in \( D_L \) has a guard. This can be done by exploring the prooftree of \( D_L \) according to all possible access paths (until for the first time a cycle is concluded) and checking for the existence of guards on cycles.

We now can prove soundness and completeness of the proof system \( \text{Reg}_{letrec}^\ast \) for the property of \( \lambda_{letrec}\)-terms to unfold to infinite \( \lambda \)-terms.

**Theorem 6.24.** \( \text{Reg}_{letrec}^\ast \) is sound and complete for the property of \( \lambda_{letrec}\)-terms to unfold to an infinite \( \lambda \)-term. That is, for every term \( L \in \text{Ter}(\lambda_{letrec}) \) the following statements are equivalent:

(i) \( L \) expresses an infinite \( \lambda \)-term.

(ii) \( \vdash_{\text{Reg}_{letrec}^\ast} ()L \).

**Proof.** For the proof of both directions of the equivalence, let \( L \in \text{Ter}(\lambda_{letrec}) \).

For showing the implication (i) \( \Rightarrow \) (ii), we argue by contraposition, and therefore assume that \( ()L \) is not derivable in \( \text{Reg}_{letrec}^\ast \). Then, while \( ()L \) is not derivable in \( \text{Reg}_{letrec}^\ast \), there is, by Proposition 6.23, (i), a derivation \( D \) in \( ug-\text{Reg}_{letrec}^\ast \) that is not guarded. It follows by Lemma 6.22, and in particular due to its item (iv), that there is a scope\(^\ast\)-delimiting strategy \( S_D \) for \( \text{Reg}^\ast \) such that \( L \) is not \( S_D\)-productive. Then it follows by Lemma 5.15, using (i) \( \Rightarrow \) (iv) there, that \( L \) does not unfold to an infinite \( \lambda \)-term.

For showing the implication (ii) \( \Rightarrow \) (i), let \( D \) be a closed derivation in \( \text{Reg}_{letrec}^\ast \) with
conclusion () \( L \). It follows by Lemma 6.22 that there is a scope\(^-\)-delimiting strategy \( S \) for \( \text{Reg} \) such that \( L \) is \( S \)-productive. Then Lemma 5.15 implies that \( L \) unfolds to an infinite \( \lambda \)-term.

\[ \text{Remark 6.25 (soundness and completeness for Reg\(_{\text{letrec}}\)). Also the proof system Reg\(_{\text{letrec}}\) can be shown to be sound and complete for the property of \( \lambda \)-\text{letrec}-terms to unfold to infinite \( \lambda \)-terms. To establish this in analogy with the route of proof we pursued here, a CRS Parse similar to Parse\(^*\) (see Definition 4.30) could be defined by replacing the rule \( (\varrho_{\text{parse}})^{\text{del}} \) by a rule \( (\varrho_{\text{del}}) \) that can compress more abstraction prefixes, similar as the rule \( (\varrho_{\text{reg}}) \) of \( \text{Reg} \) can compress more abstraction prefixes than the rule \( (\varrho_{\text{reg}}) \) of \( \text{Reg} \). Then furthermore also a lemma analogous to Lemma 5.15 can be formulated, proved, and used in a similar way.} \]

We now arrive at a theorem that states one direction of our main characterisation result (Theorem 8.12 in Section 8) that will link \( \lambda \)-\text{letrec}-expressibility to strong regularity of infinite \( \lambda \)-terms.

\[ \text{Theorem 6.26. Every } \lambda \text{-letrec-expressible, infinite } \lambda \text{-term is strongly regular.} \]

\[ \text{Proof. Let } M \text{ be an infinite } \lambda \text{-term that is expressible by a } \lambda \text{-letrec-term } L, \text{ that is, } L \rightarrow^\omega_M \text{ holds. Then by Theorem 6.24 there exists a closed derivation } D \text{ in Reg\(_{\text{letrec}}\)^* with conclusion } () L. \text{ Now Lemma 6.22 guarantees a scope}^-\text{-delimiting strategy } S_D \text{ for Reg\(_{\text{letrec}}\) such that } L \text{ is } S_D \text{-regular. Then Lemma 5.12 gives an scope}^-\text{-delimiting strategy } S_D \text{ for Reg\(_{\text{letrec}}\) such that } M = U(L) \text{ is } S_D \text{-regular. It follows that } M \text{ is strongly regular.} \]

\[ \text{7. Binding–Capturing Chains} \]

In this section we develop a characterisation for strong regularity of an infinite \( \lambda \)-term by means of a property of the ‘binding–capturing chains’ occurring in the term. This concept is related to the notions of scope and scope\(^-\) as explained informally at the start of Section 4. Binding–capturing chains occur whenever scopes overlap, and they are contained within scopes\(^-\). First we give definitions for the concepts involved: binding, capturing, and binding–capturing chains. Then we show that strong regularity is equivalent to the absence of infinite binding–capturing chains.

We will define binding and capturing as relations on the positions of a \( \lambda \)-term. Binding relates an abstraction with the occurrences of the variable it binds. If \( p \) is the position of an abstraction \( (\lambda x \ldots) \) that abstracts over \( x \) and \( q \) is the position of an occurrence of \( x \) that is bound by the abstraction, then we will write \( p \leftrightarrow q \) and say that \( p \) ‘binds’ \( q \). Capturing relates an abstraction with variable occurrences that are free in it. If \( p \) is the position of an abstraction, and \( q > p \) is the position of a variable that is free in the entire subterm at position \( p \), then we will write \( p \leftrightarrow q \) and say that \( p \) ‘captures’ \( q \). See Figure 19 for an illustration of these concepts.

When we speak of positions in \( \lambda \)-terms (and thus iCRS-terms) we act on the assumption that positions on iCRS-terms are an established concept as for example in (Ketema and Simonsen, 2011). Note, however, that we deviate slightly from the scheme there in addressing the arguments of an \text{app} by 0 and 1 instead of 1 and 2.
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Fig. 19. The term graph from Example 1.1 with its overlapping scopes (left), its nested scope’s (middle), and with indicated binding ← and capturing → links (right).

Binding–capturing chains have been used in (Endrullis et al., 2011) to study α-avoiding rewrite sequences in a rewrite calculus for μ-unfolding. They originate from the notion of ‘gripping’ due to (Melliès, 1996), and from techniques developed in (Oostrom, 1997) concerning the notion of ‘holding’ of redexes (which is shown there as being ‘parting’ for CRSs, that is, never relating two residuals of the same redex).

We now define ‘binding’ and ‘capturing’ formally as binary relations on the set of positions of infinite λ-terms.

**Definition 7.1 (binding, capturing).** For every \( M \in \text{Ter}(\lambda^\infty) \) we define the binary relations ← and → on the set \( \text{Pos}(M) \) of positions of \( M \):

(i) We say that a binder at position \( p \) binds a variable occurrence at position \( q \), symbolically \( p \leftarrow q \), if \( p \) is a binder position, and \( q \) a variable position in \( M \), and the binder at position \( p \) binds the variable occurrence at position \( q \).

(ii) We say that a variable occurrence at position \( q \) is captured by a binder at position \( p \), symbolically \( q \rightarrow p \) (and that a binder at position \( p \) captures a variable occurrence at position \( q \), symbolically \( p \leftrightarrow q \)), if \( q \) is a variable position and \( p < q \) a binder position in \( M \), and there is no binder position \( q_0 \) in \( M \) with \( p \leq q_0 \) and \( q_0 \leftarrow q \).

**Definition 7.2 (binding–capturing chain).** Let \( M \) be an infinite λ-term. A finite or infinite sequence \( \langle p_0, p_1, p_2, \ldots \rangle \) in \( \text{Pos}(M) \) is called a binding–capturing chain in \( M \) if \( p_0 \) is the position of an abstraction in \( M \), and the positions in the sequence are alternatingly linked via binding and capturing, starting with a binding: \( p_0 \leftarrow p_1 \rightarrow p_2 \leftarrow \ldots \).

Binding–capturing chains are closely related to the notion of scope and scope+. In order to establish this, we first give precise definitions of the notions of scope and scope+ in terms of an ‘in-scope’ rewrite relation on the positions of a λ-term: While the scope of a binder position \( p \) is the set of positions between \( p \) and variable positions bound at \( p \) (the positions directly reachable by a single ‘in-scope’ step), the scope+ of \( p \) is the set of positions reachable by a finite number of successive ‘in-scope’ steps.
Let $M$ be an infinite $\lambda$-term. On the set $\text{Pos}(M)$ of $M$, the \textit{in-scope} rewrite relation $\to_{sc}$ (for $M$) is defined by:

$$p \to_{sc} q \iff \begin{cases} p \text{ a binder position} \\ \land (\exists p' \in \text{Pos}(M)) \quad p \rhd p' \land p \leq q \leq p' \end{cases} \quad (\text{for all } p,q \in \text{Pos}(M))$$

where $\rhd$ denotes the reflexive closure of the binding relation $\vdash$ (for $M$). For every position $p \in \text{Pos}(M)$, the \textit{scope} of $p$ in $M$ and the \textit{scope} of $p$ in $M$ are defined as the following sets of positions in $M$:

$$\text{scope}_M(p) := \{ q \in \text{Pos}(M) \mid p \to_{sc} q \} \quad \text{scope}^*_M(p) := \{ q \in \text{Pos}(M) \mid p \to_{sc}^* q \}$$

(Note that the scopes and scope's of non-binder positions are empty sets of positions.)

Now the following proposition establishes that binding–capturing chains starting at a binder position $p$ span the space of positions of the scope of $p$.

**Proposition 7.4.** Let $M$ be an infinite $\lambda$-term. Then for all positions $p,q \in \text{Pos}(M)$ the following statements hold:

(i) $p \to_{sc} q \land q$ is a binder position $\iff (p = q$ binder position) $\lor p \rhd \cdots \rhd q$.

(ii) $p \to_{sc} q \iff (p \text{ binder pos.}) \land (3p' \in \text{Pos}(M)) \quad p (\rhd \cdots \rhd) p' \land p \leq q \leq p'$.

(iii) $\text{scope}_M(p) := \{ q \in \text{Pos}(M) \mid (p \text{ binder pos.}) \land (3p' \in \text{Pos}(M)) p (\rhd \cdots \rhd) p' \land p \leq q \leq p' \}$.

(iv) $\text{scope}^*_M(p) := \{ q \in \text{Pos}(M) \mid (p \text{ binder pos.}) \land (3p' \in \text{Pos}(M)) p (\rhd \cdots \rhd) p' \land p \leq q \leq p' \}$.

Conversely, positions between a binder position $p_0$ and a position $p_n$ on a binding–capturing chain starting at $p_n$ are in the scope of $p_0$.

**Proposition 7.5.** Let $\langle p_0, p_1, p_2, \ldots \rangle$ be a binding–capt. chain in an infinite $\lambda$-term $M$. Then it holds that $p_0 < p_2 < \ldots$, and $p_0 < p_1, p_2 < p_3, \ldots$. Furthermore, for all $q$ such that $p_0 \leq q \leq p_n$, for some $n \in \mathbb{N}$ with $p_n$ a position on the chain it holds that $q \in \text{scope}^*_M(p_0)$.

In order to study the relationship between rewrite sequences in $\text{Reg}^*$ and binding–capturing chains we first introduce a position-annotated variant of $\text{Reg}^*$.

Here the idea is that, when a prefixed term $(\lambda y_1 \ldots y_n)N$ is obtained as a generated subterm of an infinite $\lambda$-term $M$ by a $\rightarrow_{\text{reg}}$ or $\rightarrow_{\text{reg}^*}$ rewrite sequence $\tau$ on $M$, then in the position-annotated rewrite system a prefixed term $(\lambda y_1, \ldots, y_n)_{p_1,\ldots,p_n}^q N$ is obtained by an annotated version $\tau^\text{pos}$ of the rewrite sequence $\tau$ such that: the positions $p_1, \ldots, p_n$ are the positions in (the original $\lambda$-term) $M$ from which the bindings $\lambda y_1, \ldots, \lambda y_n$ in the abstraction prefix descend, and $q$ is the position in $M$ of the body $N$ of the subterm generated by $\tau$.

In $\text{Ter}(\lambda^n)$ we consider the following rewrite rules in informal notation:

- $(\varepsilon_{\text{pos}}^0) : \quad (\lambda x_1 \ldots x_n)_{p_1,\ldots,p_n}^q M_0 M_1 \to (\lambda x_1 \ldots x_n)_{p_1,\ldots,p_n}^{q+i} M_i \quad (i \in \{0,1\})$
- $(\varepsilon_{\text{pos}}^1) : \quad (\lambda x_1 \ldots x_n)_{p_1,\ldots,p_n}^q \lambda y M_0 \to (\lambda x_1 \ldots x_n y)_{p_1,\ldots,p_n,q}^{q+0} M_0$
- $(\varepsilon_{\text{pos}}^5) : \quad (\lambda x_1 \ldots x_{n+1})_{p_1,\ldots,p_{n+4}}^q M_0 \to (\lambda x_1 \ldots x_n)_{p_1,\ldots,p_n}^q M_0 \quad (\text{if the binding } \lambda x_{n+1} \text{ is vacuous})$
Conversely, every rewrite sequence $\langle \tau \rangle$ in $\text{Reg}_{\text{pos}}$ preserves eagariness/laziness of rewrite sequences.

Definition 7.6 (position-annotated variants $\text{Reg}_{\text{pos}}$ and $\text{Reg}^*_{\text{pos}}$). The CRS-signature for $(\lambda)_{\text{pos}}$, the $\lambda$-calculus with position-annotated abstraction prefixes is given by $\Sigma_{(\lambda)_{\text{pos}}} = \Sigma_{\lambda} \cup \{ \text{pre}_{\langle p_1, \ldots, p_n \rangle}^q | p_1, \ldots, p_n, q \in \{0,1\}^* \}$ where all of the function symbols $\text{pre}_{\langle p_1, \ldots, p_n \rangle}^q$ are unary. We consider the following CRS-rules over $\Sigma_{(\lambda)_{\text{pos}}}$:

$$\begin{align*}
\varepsilon_{\text{pos}}^{\text{del}} & : (\lambda x_1 \ldots x_{n+1})_{p_1, \ldots, p_{n+1}} \rightarrow (\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})_{p_1, \ldots, p_i-1, p_{i+1} \ldots p_{n+1}} M_0 \\
& \quad \text{(if the binding } \lambda x_i \text{ is vacuous)}
\end{align*}$$

Note that the change of the term-body position in a $\lambda$-decomposition step is motivated by the underlying CRS-notation for terms in $(\lambda^n)$: when a term $\text{abs}([y] M_0)$ representing a $\lambda$-abstraction starts at position $q$, then its binding is declared at position $q_0$, and its body $M_0$ starts at position $q_0$.

By $\text{Reg}^-_{\text{pos}}$ we denote the CRS with the rules $\varepsilon_{\text{pos}}^{\text{del}}$ and $\varepsilon_{\text{pos}}^\lambda$. By $\text{Reg}_{\text{pos}}$ ($\text{Reg}^*_{\text{pos}}$) we denote the CRS consisting of all the above rules except the rule $\varepsilon_{\text{pos}}^{\text{del}}$ ($\varepsilon_{\text{pos}}^\lambda$).

By $\text{Reg}^-_{\text{pos}}$, $\text{Reg}_{\text{pos}}$, and $\text{Reg}^*_{\text{pos}}$, we denote the infinite abstract rewriting systems (ARSs) induced by the iCRSs derived from $\text{Reg}^-_{\text{pos}}$, $\text{Reg}_{\text{pos}}$, $\text{Reg}^*$, restricted to position-annotated terms in $\text{Ter}((\lambda^n))$.

By $\text{drop}_{\text{pos}}$ we denote an operation that drops the position annotations in CRS-terms.

Proposition 7.7. The following two statements hold:

(i) Every rewrite sequence:

$$\tau : (\lambda \vec{x}_0) M_0 \rightarrow_r (\lambda \vec{x}_1) M_1 \rightarrow_r \ldots \rightarrow_r (\lambda \vec{x}_n) M_n \quad (7.9)$$

(with $r \in \{\text{reg}^-, \text{reg}, \text{reg}^+\}$) in $\text{Reg}^-$, $\text{Reg}$, or $\text{Reg}^+$ can stepwisely be transformed (lifted), for given $q_0 \in \mathbb{N}^*$ and $\vec{p}_0 \in \mathbb{N}^*$ with $[\vec{p}_0] = [\vec{x}_0]$, by adding these and appropriate further position annotations $q_1, \ldots, q_n \in \mathbb{N}^*$ and $\vec{p}_1, \ldots, \vec{p}_n \in \mathbb{N}^*$, to a rewrite sequence:

$$\tau_{\text{pos}} : (\lambda \vec{x}_0)_{q_0, \vec{p}_0} M_0 \rightarrow_r (\lambda \vec{x}_1)_{q_1, \vec{p}_1} M_1 \rightarrow_r \ldots \rightarrow_r (\lambda \vec{x}_n)_{q_n, \vec{p}_n} M_n \quad (7.10)$$

(with $r \in \{\text{reg}^-, \text{reg}, \text{reg}^+\}$) in $\text{Reg}^-_{\text{pos}}$, $\text{Reg}_{\text{pos}}$, or $\text{Reg}^*_{\text{pos}}$, accordingly, such that the result of dropping the position annotations in the prefix of $\tau$ is again $\tau$.

(ii) Conversely, every rewrite sequence $\xi$ in $\text{Reg}^-_{\text{pos}}$, $\text{Reg}_{\text{pos}}$, or $\text{Reg}^*_{\text{pos}}$ of the form (7.10) (with $r \in \{\text{reg}^-, \text{reg}, \text{reg}^+\}$) can stepwisely be transformed, by dropping the position annotations in the prefix, to a rewrite sequence $\xi$ of the form (7.10) (with $r \in \{\text{reg}^-, \text{reg}, \text{reg}^+\}$) in $\text{Reg}^-$, $\text{Reg}$, or $\text{Reg}^*$, respectively.

The transformations in (i) and (ii) preserve eagerness/laziness of rewrite sequences.

As a direct consequence we obtain, for the eager and lazy scope-delimiting (scope$^*$-
delimiting) strategies, the following direct correspondence between generated subterms in the ARS \( \text{Reg} (\text{Reg}^*) \) and in the position-annotated version \( \text{Reg}_{pos} (\text{Reg}^*_{pos}) \).

**Proposition 7.8.** For all infinite \( \lambda \)-terms \( M \) it holds:

(i) \( ST^pS(M) = \text{drop}^{pos} \left(ST^pS(M) \right) \) for the strategies \( S \in \{ S^{\text{tag}}, S^{\text{lazy}} \} \) on \( \text{Reg}, \text{Reg}^*_{pos} \).

(ii) \( ST^pS(M) = \text{drop}^{pos} \left(ST^pS(M) \right) \) for the strategies \( S \in \{ S^{\text{tag}}, S^{\text{case}} \} \) on \( \text{Reg}, \text{Reg}^*_{pos} \).

The proposition below characterizes the binding relation \( \rightsquigarrow \) and the capturing relation \( \rightsquigarrow \) on the positions of an infinite term \( M \) with the help of rewrite sequences with respect to \( \rightsquigarrow_{\text{reg}} \) on \( \lambda^* M \) in \( \text{Reg}^*_{pos} \) down to ‘variable occurrences’ \( (\lambda \bar{x})^q_{i} \bar{x}_i \) in \( M \).

**Proposition 7.9.** For all \( M \in \text{Ter}(\lambda^*) \) and positions \( p, q \in \text{Pos}(M) \) it holds:

\[ p \rightsquigarrow q \iff \text{there is a rewrite sequence} \ (\lambda^* M) \rightarrow_{\text{reg}} (\lambda x_1 \ldots x_n)^q_{p_1 \ldots p_n} x_i \text{ such that} \ p = p_i \]

\[ q \rightarrow p \iff \text{there is a rewrite sequence} \ (\lambda^* M) \rightarrow_{\text{reg}} (\lambda x_1 \ldots x_n)^q_{p_1 \ldots p_n} x_i \text{ such that} \ p \in \{ p_{i+1}, \ldots, p_n \} \]

The following lemmas describe the close relationship between, on the one hand, binding–capturing chains in an infinite \( \lambda \)-term \( M \), and on the other hand, \( \rightarrow_{\text{reg}^*} \)–rewrite sequences on \( \lambda^* M \) in \( \text{Reg}^*_{pos} \) that are guided by the eager scope\(^*\)-delimiting strategy.

**Lemma 7.10 (binding–capturing chains).** For all \( M \in \text{Ter}^\omega(\lambda^*) \) it holds:

(i) If \( \lambda^* M \rightarrow_{S^{\text{tag}}} (\lambda x_1 \ldots x_n)^q_{p_1 \ldots p_n} N \) in \( \text{Reg}^* \), then there exist \( q_2, \ldots, q_n \in \text{Pos}(M) \) such that \( p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \ldots \rightarrow q_n \rightarrow p_n \).

(ii) If \( p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \ldots \rightarrow q_n \rightarrow p_n \) is a binding–capturing chain in \( M \), then there exist \( r_1, \ldots, r_n, s \in \text{Pos}(M) \) such that \( \lambda^* M \rightarrow_{S^{\text{tag}}} (\lambda x_1 \ldots x_n)^s_{r_1 \ldots r_n} N \) and furthermore \( p_1, \ldots, p_n \in \{ r_1, \ldots, r_n \} \) such that \( p_1 < p_2 < \ldots < p_n \).

**Lemma 7.11.** Let \( M \) be an infinite \( \lambda \)-term such that \( \lambda^* M \rightarrow_{S^{\text{tag}}} (\lambda x_1 \ldots x_n) N \). Then \( M \) contains a binding–capturing chain of length \( n - 1 \).

**Proof.** By Proposition 7.7, (i), the assumed rewrite sequence \( \lambda^* M \rightarrow_{S^{\text{tag}}} (\lambda x_1 \ldots x_n) N \) in \( \text{Reg}^* \) can be lifted to a rewrite sequence \( \lambda^* M \rightarrow_{S^{\text{tag}}} (\lambda x_1 \ldots x_n)^q_{p_1 \ldots p_n} N \) in \( \text{Reg}^*_{pos} \). Then by Lemma 7.10, (i), there exists a binding–capturing chain of length \( n - 1 \).

The notion of scope and scope\(^*\) helps to understand the relationship between binding–capturing chains and rewrite sequences in \( \text{Reg}^* \). A binding–capturing chain corresponds to the overlap of scopes, or in other words the nesting of scope\(^*\)’s. An infinite binding–capturing chain thus corresponds to an infinitely deep nesting of scope\(^*\)’s and therefore to an unrestricted growth of the prefix in certain rewriting sequences in \( \text{Reg}^* \).

**Lemma 7.12 (infinite binding–capturing chains).** Let \( M \) be an infinite \( \lambda \)-term, and let \( \tau \) be an infinite rewrite sequence in \( \text{Reg} \) w.r.t. the eager scope\(^*\)-delimiting strategy \( S^{\text{tag}} \): 

\[ \tau : \ (\lambda^* M) \rightarrow (\lambda \bar{x}_0) M_0 \rightarrow_{S^{\text{tag}}} (\lambda \bar{x}_1) M_1 \rightarrow_{S^{\text{tag}}} \ldots \rightarrow_{S^{\text{tag}}} (\lambda \bar{x}_i) M_i \rightarrow_{S^{\text{tag}}} \ldots \]  

(7.11)

Furthermore suppose that for \( p : \mathbb{N} \rightarrow \mathbb{N}, i \mapsto p(i) := |\bar{x}_i| \), the prefix length function associated with \( \tau \), there exists a lower bound \( l : \mathbb{N} \rightarrow \mathbb{N} \) such that \( l \) is non-decreasing, and \( \lim_{n \rightarrow \infty} l(b(n)) = \infty \). Then there exists an infinite binding–capturing chain in \( M \).
Proof. Let $M$, $\tau$, $p$, $lb$ as in the assumption of the lemma. We first note that by Proposition 7.7, (i), the rewrite sequence $\tau$ can be lifted to one with position annotations:

$$
\tau^\text{pos} : (\lambda \bar{x}_0)^{\bar{y}} M_0 \Rightarrow_{\text{seq}} (\lambda \bar{x}_1)^{\bar{y}_1} M_1 \Rightarrow_{\text{seq}} \ldots \Rightarrow_{\text{seq}} (\lambda \bar{x}_i)^{\bar{y}_i} M_i \Rightarrow_{\text{seq}} \ldots
$$

(7.12)

where, for all $i \in \mathbb{N}$, $q_i$ are positions and $\bar{p}_i = \langle p_1, \ldots, p_{m_i} \rangle$ vectors of positions, with $m_i \in \mathbb{N}$.

Next we define the function:

$$
st : \mathbb{N} \to \mathbb{N}, \ l \mapsto st(l) := \min\{i \mid lb(i) \geq l\}
$$

which is well-defined, since $\lim_{n \to \infty} lb(n) = \infty$. It describes a prefix stabilization property: for every $l \in \mathbb{N}$, it gives the first index $i = st(l)$ with the property that the prefix of $(\lambda \bar{x}_j) M_j$ contains more than $l$ abstractions, and (since $lb$ is non-decreasing, and a lower bound for $p$) that from $i$ onwards the $l$-th abstraction never disappears again, for $j \geq i$, in terms $(\lambda \bar{x}_j) M_j$ that follow in $\tau$ as well as in $\tau^\text{pos}$. Furthermore, $st$ is non-decreasing, as an easy consequence of its definition, and unbounded: if $st$ were bounded by $M \in \mathbb{N}$, then $\forall l \in \mathbb{N} \exists i \in \mathbb{N}, i \leq M \land lb(i) \geq l$ would follow, which cannot be the case since { $lb(0), \ldots, lb(M)$ } is a finite set. By non-decreasingness and unboundedness it also follows that $\lim_{n \to \infty} st(n) = \infty$.

So when the rewrite sequence $\tau^\text{pos}$ is split into segments indicated in:

$$
(\lambda \bar{x}_j)^{\bar{y}} M \Rightarrow_{\text{seq}} \ldots \Rightarrow_{\text{seq}} (\lambda \bar{x}_j)^{\bar{y}_1} M_{st(1)} \Rightarrow_{\text{seq}} \ldots
$$

then it follows that all terms of the sequence after $(\lambda \bar{x}_j)^{\bar{y}_1} M_{st(1)}$ have an abstraction prefix of length greater or equal to $i$, for all $i \in \mathbb{N}$.

Now note that in a step $(\lambda \bar{x})^{\bar{y}} (p_1, \ldots, p_n) P \Rightarrow (\lambda \bar{x})^{\bar{y}_1}(p_1', \ldots, p_n') P'$ in $\text{Reg}_\text{pos}$ that does not shorten the abstraction prefix it holds that $n \leq n'$ and $p_1 = p_1', \ldots, p_n = p_n$, that is, positions in the vector in the subscript of the abstraction prefix are preserved. As a consequence it follows for the rewrite sequence $\tau^\text{pos}$ that, for all $i \in \mathbb{N}$ and $j > i$, the position vector $\bar{p}_{st(j)}$ in the term $(\lambda \bar{x}_{st(i)})^{\bar{y}_{st(i)}} M_{st(i)}$ is of the following form:

$$
\bar{p}_{st(j)} = \langle p_{1, st(i)}, \ldots, p_{i, st(i)}, p_{j, st(j)}, \ldots, p_{j, m_j} \rangle
$$

This implies furthermore that for all $i \in \mathbb{N}$:

$$
\bar{p}_{st(i)} = \langle p_{1, st(1)}, p_{2, st(2)}, \ldots, p_{i, st(i)}, \ldots, p_{i, m_i} \rangle
$$

Then Lemma 7.10, (i), implies the existence of positions $q_2, q_3, \ldots$ such that:

$$
p_i, st(1) \leftarrow q_2 \ast p_{2, st(2)} \leftarrow q_3 \ast \ast, \ldots \ast, p_i, st(i) \leftarrow q_{i+1} \ast p_{i+1, st(i+1)} \leftarrow \ast, \ldots
$$

and thereby, an infinite binding–capturing chain in $M$.

Now we formulate and prove the main theorem of this section, which applies the concept of binding–capturing chain to pin down, among all infinite $\lambda$-terms that are regular, those that are strongly regular.

**Theorem 7.13.** A regular infinite $\lambda$-term is strongly regular if and only if it contains only finite binding–capturing chains.

By adding the statement of Proposition 4.22, (i), we obtain the following accentuation.
Corollary 7.14. A strongly regular \( \lambda \)-term is regular, and contains only finitely many binding-capturing chains. Conversely, a regular \( \lambda \)-term is strongly regular if it contains only finite binding-capturing chains.

Proof of Theorem 7.13. Let \( M \) be an infinite \( \lambda \)-term that is regular.

For showing the implication "\( \Rightarrow \)" we assume that \( M \) is also strongly regular. Then there exists a scope\(^{+}\)-delimiting strategy \( S \) such that \( ST^+_{S_{\text{eag}}} (M) \) is finite. By Proposition 4.23, (i) it follows that then also \( ST^+_{S_{\text{eag}}}(M) \) is finite for the eager scope\(^{+}\)-delimiting strategy \( S_{\text{eag}} \) in \( \text{Reg}^+ \). Now let \( n \) be the longest abstraction prefix of a term in \( ST^+_{S_{\text{eag}}}(M) \). Then it follows by Lemma 7.11 that the length of every binding-capturing chain in \( M \) is bounded by \( n - 1 \). Hence \( M \) only contains finite binding-capturing chains.

In the rest of this proof, we establish the implication "\( \Leftarrow \)" in the statement of the theorem. For this we argue indirectly: assuming that \( M \) is not strongly regular, we show the existence of an infinite binding-capturing chain in \( M \).

So suppose that \( M \) is not strongly regular. Then for all scope\(^{+}\)-delimiting strategies \( S \) in \( \text{Reg}^+ \) it holds that \( ST^+_{S}(M) \) is infinite. This means that in particular \( ST^+_{S_{\text{eag}}}(M) \) is infinite for the eager scope-delimiting strategy \( S_{\text{eag}} \) on \( \text{Reg}^+ \). It follows that the number of \( \rightarrow_{S_{\text{eag}}} \)-reducts, and hence the induced sub-ARS (\( (M \rightarrow_{S_{\text{eag}}}^*) (M) \) in \( \text{Reg}^+ \) is infinite. Since \( \rightarrow_{S_{\text{eag}}} \) on \( \text{Reg}^+ \) has branching degree \( \leq 2 \) (branching actually only happens at sources of \( \rightarrow_{\alpha} \)-steps), it follows by König’s Lemma that there exists an infinite rewrite sequence:

\[
\tau : (\lambda \bar{x}_0) M_0 \rightarrow_{S_{\text{eag}}} (\lambda \bar{x}_1) M_1 \rightarrow_{S_{\text{eag}}} \ldots \rightarrow_{S_{\text{eag}}} (\lambda \bar{x}_i) M_i \rightarrow_{S_{\text{eag}}} \ldots
\]

in \( \text{Reg}^+ \) that passes through distinct terms. By Lemma 4.12, (i), this rewrite sequence projects to a rewrite sequence:

\[
\breve{\tau} : (\lambda \bar{x}_0') M_0 \rightarrow_{S_{\text{eag}}} (\lambda \bar{x}_1') M_1 \rightarrow_{S_{\text{eag}}} \ldots \rightarrow_{S_{\text{eag}}} (\lambda \bar{x}_i') M_i \rightarrow_{S_{\text{eag}}} \ldots
\]

in \( \text{Reg} \) in the sense that:

\[
(\lambda \bar{x}_i) M_i \rightarrow_{\text{del}} (\lambda \bar{x}_i') M_i \quad \text{(for all } i \in \mathbb{N}) ;
\]

note that, in the terms, the projection merely shortens the length of the abstraction prefix. Since \( M \) is regular, \( ST^+_{S_{\text{eag}}}(M) \) is finite by Proposition 4.23, (i), and hence it follows that only finitely many terms occur in \( \breve{\tau} \).

Now we will use this contrast with \( \tau \), and the fact that the terms of \( \tau \) project to terms in \( \breve{\tau} \) via \( \rightarrow_{\text{del}} \)-prefix compression rewrite sequences, to show that the prefix lengths in terms of \( \tau \) are unbounded, and stronger still, that these lengths actually tend to infinity.

More precisely, we show the following:

\[
(\forall l \in \mathbb{N}) \exists i_0 \in \mathbb{N} (\forall i \geq i_0) \lceil |\bar{x}_i| \rceil \geq l.
\]

Suppose that this statement does not hold. Then there exists \( l_0 \in \mathbb{N} \) such that \( |\bar{x}_i| < l_0 \) for infinitely many \( i \in \mathbb{N} \). This means that there is an increasing sequence \( i_0 < i_1 < i_2 < i_3 < \ldots \) in \( \mathbb{N} \) such that:

\[
S := \{(\lambda \bar{x}_{i_j}) M_{i_j} | j \in \mathbb{N} \} \text{ is infinite}
\]

for all \((\lambda \bar{x}_{i_j}) M_{i_j} \in S \): \( |\bar{x}_{i_j}| < l_0 \)
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(S is infinite since the terms on τ are distinct). On the other hand we have:

\[ T := \left\{ (\lambda x_j^i) M_{i,j} \mid j \in \mathbb{N} \right\} \subseteq ST_{\text{Reg}}(M) \text{ is finite} \]

(7.18)
because M is regular. However, since every term in S has a \( \Rightarrow_{\text{del}} \)-reduct in T due to (7.14), as well as an abstraction prefix of a length bounded by \( l_0 \), it follows by Proposition 4.9, (ii), that S also has to be finite, conflicting with (7.16). We have reached a contradiction, and thereby established (7.15).

Now we are able to define a lower bound on the lengths of the prefixes in τ that fulfills the requirements of Lemma 7.12. We define the function:

\[ lb : \mathbb{N} \to \mathbb{N}, \ n \mapsto lb(n) := \min \{ |x_n'| \mid n' \geq n \} \]

Its definition guarantees that \( lb \) is a lower bound on the prefix lengths in τ, and that \( lb \) is non-decreasing. Furthermore also \( \lim_{n \to \infty} lb(n) = \infty \) follows by non-decreasingness, in addition to unboundedness of \( lb \): for arbitrary \( l \in \mathbb{N} \), by (7.15) there exists \( n_0 \in \mathbb{N} \) such that \( |x_n| \geq l \) holds for all \( n \in \mathbb{N}, \ n \geq n_0 \); this entails \( lb(n_0) \geq l \).

Now since \( M, \tau \), together with \( lb \) as defined above, satisfy the assumptions of Lemma 7.12, this lemma can be applied, yielding an infinite binding–capturing chain in \( M \).

Example 7.15. The infinite \( \lambda \)-term from Example 1.1 with a representation as a higher-order recursive program scheme in Example 4.25, which was recognised there to be regular but not strongly regular, possesses an infinite binding–capturing chain as indicated on the right in Figure 19.

8. Expressibility by terms of the \( \lambda \)-calculus with letrec

In this section we finish the proof of our main characterisation result: we prove that every strongly regular \( \lambda \)-term is \( \lambda \text{-letrec} \)-expressible. For this purpose we introduce an annotated variant of one of the proof systems for strongly regular infinite \( \lambda \)-terms. We show that every closed derivation in \( \text{Reg}_0^+ \) with conclusion \(( \) \( M \), which witnesses that \( M \) is strongly regular, can be annotated, by adding appropriate \( \lambda \text{-letrec} \)-terms to each prefixed term in the derivation, into a derivation in the annotated system with conclusion \(( \) \( L : M \) such that the \( \lambda \text{-letrec} \)-term annotation \( L \) expresses the infinite \( \lambda \)-term \( M \). We show the correctness of this construction by transforming the derivation in the annotated proof system into a derivation in the proof system \( \text{Reg}_0^* \) with conclusion \(( \) \( L : M \), and then drawing upon the soundness of \( \text{Reg}_0^* \) with respect to equality of strongly regular infinite \( \lambda \)-terms.

We start by introducing a variant of the proof system \( \text{Reg}_0^* \) in which the formulas are closed, prefixed, \( \lambda \text{-letrec} \)-term-annotated, infinite \( \lambda \)-terms.

Definition 8.1 (the proof system \( \text{ann-Reg}_0^* \)). The formulas of the proof system \( \text{ann-Reg}_0^* \) are closed expressions of the form \(( \lambda x) L : M \) with \( x \) a variable prefix vector, \( \lambda x.L \) a \( \lambda \text{-letrec} \)-term, and \( \lambda x.M \) a \( \lambda \)-term. The axioms and rules of \( \text{ann-Reg}_0^* \) are annotated versions of the axioms and rules of the proof system \( \text{Reg}_0^* \) from Definition 6.3 and Figure 12, and are displayed in Figure 20.

Remark 8.2. As for an example that illustrates why we have chosen to formulate an annotated version only of the proof system \( \text{Reg}_0^* \), but not of \( \text{Reg}^* \), please see Example 8.5.
which guarantees that an annotated version $\hat{D}$ of an open assumptions) in $\text{ann-Reg}^*$ can be annotated to a derivation $D$ if it has to be shown that a derivation $D$ with assumptions ($\ premise (\base_case, axioms (\divides.alt0$ on the depth of infinite $\lambda$-terms, a version of $\text{Reg}^*$ with $\lambda_{\text{letrec}}$-terms as annotations.

The following proposition is a statement that is entirely analogous to Proposition 6.5.

**Proposition 8.3.** For all for all instances $\iota$ of the rule (FIN) in a derivation $D$ (possibly with open assumptions) in $\text{ann-Reg}^*$ it holds: every thread from $\iota$ upwards to a marked assumption that is discharged at $\iota$ passes at least one instance of a rule ($\lambda$) or (\@).

The lemma below states a straightforward connection between derivations in $\text{Reg}^*$ and derivations in its annotated version $\text{ann-Reg}^*$.

**Lemma 8.4 (from $\text{Reg}^*$- to $\text{ann-Reg}^*$-derivations, and back).** The following transformations are possible between derivations in $\text{Reg}^*$ and derivations in $\text{ann-Reg}^*$:

(i) Every derivation $D$ in $\text{Reg}^*$ with conclusion $\lambda x y : M$ can be transformed into a derivation $\hat{D}$ in $\text{ann-Reg}^*$ with conclusion $\lambda x y : L : M$ such that there is a bijective correspondence between marked assumptions $((\lambda y y) M)^w$ in $D$ and marked assumptions $((\lambda y y) M)^w$ in $\hat{D}$. (As a consequence, $\hat{D}$ is a closed derivation if $D$ is closed.) More precisely, $\hat{D}$ can be obtained from $D$ by replacing every term occurrence $\lambda y y N$ by an occurrence of $\lambda y y P : N$ for a prefixed $\lambda_{\text{letrec}}$-term $\lambda y y P$ with the property that every prefix variable $y_i$ bound in $P$ is also bound in $N$. Thereby occurrences of marked assumptions and axioms $\lambda y y 0$ in $D$ give rise to occurrences of marked assumptions and axioms $\lambda y y 0$ in $\hat{D}$, respectively; instances of the $\text{Reg}^*$-rules $\lambda$, $\@$, $\S$, and (FIN) in $D$ give rise to instances of $\text{ann-Reg}^*$-rules $\lambda$, $\@$, $\S$, and (FIN) in $\hat{D}$, respectively.

(ii) From every closed derivation $D$ in $\text{ann-Reg}^*$ with conclusion $\lambda x y : L : M$ a closed derivation $\hat{D}$ in $\text{Reg}^*$ with conclusion $\lambda x y : M$ can be obtained by dropping the annotations with $\lambda_{\text{letrec}}$-terms.

**Proof.** Statement (i) of the lemma can be established through a proof by induction on the depth $|D|$ of a derivation $D$ in $\text{Reg}^*$ with possibly open assumptions. In the base case, axioms (0) of $\text{Reg}^*$ are annotated to axioms (0) of $\text{ann-Reg}^*$, and marked assumptions $((\lambda y y) N)^w$ in $\text{Reg}^*$ to marked assumptions $((\lambda y y) c_{u : N})^w$. In the induction step it has to be shown that a derivation $D$ in $\text{Reg}^*$ with immediate subderivation $D_0$ can be annotated to a derivation $\hat{D}$ in $\text{ann-Reg}^*$, using the induction hypothesis which guarantees that an annotated version $\hat{D}_0$ of $D_0$ has already been obtained. Then
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for obtaining $D$ from $D_0$ the fact is used that the rules in $\text{ann-Reg}_0^\ast$ uniquely determine the annotation in the conclusion of an instance once the annotation(s) in the premise(s) (and in the case of (FIX) additionally the annotation markers used in the assumptions that are discharged) are given. In order to establish that instances of $S$ in $D$ give rise to corresponding instances of $S$ in $\hat{D}$, the part of the induction hypothesis is used which guarantees that the $\lambda_{\text{letrec}}$-term annotation in the premise contains not more variable bindings than the infinite $\lambda$-term it annotates.

Statement (ii) of the lemma is a consequence of the fact that, by dropping the $\lambda_{\text{letrec}}$-term-annotations, every instance of a rule of $\text{ann-Reg}_0^\ast$ give rise to an instance of the corresponding rule in $\text{Reg}_0^\ast$. Formally the statement can again be established by induction on the depth of derivations in $\text{ann-Reg}_0^\ast$.

Example 8.5. The derivation $D_l$ in $\text{Reg}_0^\ast$ from Example 6.6 on the left can be annotated, as described by Lemma 8.4, (ii), to obtain the following derivation $\hat{D}_l$ in $\text{ann-Reg}_0^\ast$:

\[
\frac{((\lambda x. c u : M) u) : \lambda y. M x y}{((\lambda x. c u : M) u) : \lambda y. M x y} \quad (\lambda x. c u : M) u : M \quad S \quad (\lambda x. x : x) \quad 0 \quad @ 
\frac{(\lambda x. c u. x : M x) u}{(\lambda x. c u. x : M x) u : M x} \quad S \quad (\lambda x. y : y) \quad 0 \quad @ 
\frac{(\lambda x. y : y) y : y}{(\lambda x. y : y) y : y} \quad \lambda 
\frac{(\lambda x. c u. x : M x) u : M x \lambda}{(\lambda x. c u. x : M x) u : M x \lambda} \quad (\lambda x. c u. x : M x) u : M x \lambda
\]

Note that the term in the conclusion, which has been extracted by the annotation procedure, is actually the same as the $\lambda_{\text{letrec}}$-term $\text{letrec } u = \lambda y. u x y \text{ in } f$ which was used in Example 6.6 to define $M$ as its infinite unfolding.

Furthermore note that, in a variant of $\text{ann-Reg}_0^\ast$ in which the ‘$\text{Reg}_0^\ast$-addition’ (concerning abstraction prefix lengths) to the side-condition of (FIX) is dropped, the derivation $D_r$ in Example 6.6 on the right could be annotated to obtain the following prooftree $\hat{D}_r$:

\[
\frac{((\lambda x. u : \lambda y. M x y) u) : \lambda y. M x y}{(\lambda x. u : \lambda y. M x y) u : \lambda y. M x y} \quad \lambda 
\frac{(\lambda x. u : \lambda y. M x y) u : \lambda y. M x y}{(\lambda x. u : \lambda y. M x y) u : \lambda y. M x y} \quad \lambda
\]

Observe that, equally as was the case for $D_r$, also in $\hat{D}_r$ there occurs, on the thread between the marked assumption at the top and the rule instance $\iota$ at which this assumption is discharged, a formula, namely $() u : M$, that has a shorter abstraction prefix than the
Theorem 8.11. But as an intermediary proof system that will allow us to use results about version of assumption-discharging rule in the system, (\text{FIX}) in
\begin{align*}
\{ (\lambda \vec{x}) \ c_{f_1} : M_1 \} &\Rightarrow \{ (\lambda \vec{x}) \ c_{f_n} : M_n \} \\
D_j &\Rightarrow D_{n+1} \\
\{ \ldots \ldots \ldots \} &\Rightarrow (\lambda \vec{x}) \ (\text{letrec} \ f_1 = L_1 \ldots f_n = L_n \in M_{n+1}) : M_{n+1}
\end{align*}
\(\text{FIX}_{\text{letrec}}\)

where \(c_{f_1}, \ldots, c_{f_n}\) are distinct constants fresh for \(L_1, \ldots, L_{n+1}\), and substitutions \(L_i[\vec{x} := \vec{\xi}_i]\) stands short for \(L_i[\vec{x} := \vec{\xi}_f, \ldots, \vec{x} := \vec{c}_{f_n}]\).

\textbf{side-conditions:} \(|y| \geq |\vec{x}|\) holds for the prefix length of every \((\lambda \vec{y})\) \(N\) on a thread in \(D_j\) for \(1 \leq j \leq n + 1\) from an open assumptions \((\lambda \vec{x})c_{f_i})^{\omega_i}\) downwards; for bottommost instances: the arising derivation is guarded on access path cycles.

\textbf{Proof.}

\(\text{ann-Reg}^*\) formula in the premise and conclusion of \(\iota\) as well as in the assumption. Thus \(\iota\) is not an instance of the rule \((\text{FIX})\) in \(\text{ann-Reg}^*_0\).

Furthermore note that the \(\lambda_{\text{letrec}}\)-term extracted by \(\bar{D}_r\) does not unfold to \(M\), and hence does not express \(M\). This example shows that the side-condition on instances of \((\text{FIX})\) in \(\text{ann-Reg}^*_0\) cannot be weakened to the form used for the rule \((\text{FIX})\) in \(\text{Reg}^*\) when the aim is to extract a \(\lambda_{\text{letrec}}\)-term that unfolds to the infinite \(\lambda\)-term in the conclusion.

The central property of the proof system \(\text{ann-Reg}^*_0\) still remains to be shown: that the \(\lambda_{\text{letrec}}\)-terms in the conclusion of a derivation in this system does actually unfold to the infinite \(\lambda\)-term in the conclusion. This will be established below in Lemma 8.10 and Theorem 8.11. But as an intermediary proof system that will allow us to use results about the proof system \(\text{Reg}^*_{\text{letrec}}\) from Section 6, we also introduce an annotated version of the rule \text{letrec} in \(\text{Reg}^*_{\text{letrec}}\) and an according annotated proof system.

\textbf{Definition 8.6 (the proof system \text{ann-Reg}^*_{\text{letrec}}).}

The proof system \(\text{ann-Reg}^*_{\text{letrec}}\) arises from \(\text{Reg}^*_0\) by replacing the rule \((\text{FIX})\) by the rule \((\text{FIX}_{\text{letrec}})\) in Figure 21, an annotated version of the rule \((\text{FIX}_{\text{letrec}})\) from Definition 6.18 and Figure 18. The side-condition on bottommost instances of \((\text{FIX}_{\text{letrec}})\) to be guarded on access path cycles is analogous as explained in Definition 6.18.

\textbf{Proposition 8.7 (from \text{ann-Reg}^*_{\text{letrec}} \text{- to \text{Reg}^*_{\text{letrec}}-derivations}).}

Let \(D\) be a closed derivation in \(\text{ann-Reg}^*_{\text{letrec}}\) with conclusion \((\lambda \iota) \ M\). Then a closed derivation \(\bar{D}\) in \(\text{Reg}^*_{\text{letrec}}\) with conclusion \((\lambda \iota) \ M\) can be obtained by removing the infinite \(\lambda\)-terms in \(D\) while keeping the \(\lambda_{\text{letrec}}\)-term-annotations.

\textbf{Proposition 8.8 (from \text{ann-Reg}^*_0 \text{- to \text{ann-Reg}^*_{\text{letrec}}-derivations}).}

Every derivation \(D\) in \(\text{ann-Reg}^*_0\) can be transformed into a derivation \(D'\) in \(\text{ann-Reg}^*_{\text{letrec}}\) with the same conclusion and with the same open assumption classes.

\textbf{Proof.} First note that the \(\text{ann-Reg}^*_0\) and \(\text{ann-Reg}^*_{\text{letrec}}\) differ only by the specific version of assumption-discharging rule in the system, \((\text{FIX})\) in \(\text{ann-Reg}^*_0\) and \((\text{FIX}_{\text{letrec}})\) in \(\text{ann-Reg}^*_{\text{letrec}}\). For showing the proposition, let \(D\) be a derivation in \(\text{ann-Reg}^*_0\).

We define a prooftree \(D'\), (intended to be a derivation in \(\text{ann-Reg}^*_{\text{letrec}}\)) by repeatedly...
replacing topmost occurrences of (FIX) at the bottom of subderivations of the form as depicted in Figure 20, by simulating subderivations of the form:

\[
\begin{align*}
& \text{letrec } u = L \in u : M \\
& \text{FIX}_\text{letrec, } u
\end{align*}
\]

until all occurrences of instances of (FIX) have been replaced by instances of (FIX\text{letrec}). The result is a proof-tree with axioms and rules of ann-Reg\text{letrec} + (FIX\text{letrec}), with the same conclusion and the same classes of open assumptions as \(D\), but in which rule instances carrying the label (FIX\text{letrec}) might actually be instances of (FIX\text{letrec}), unless actually proven (as will be done below) to be instances of (FIX\text{letrec}).

Now first note that, due to the form of the introduced instances of (FIX\text{letrec}), every formula occurrence in \(D\) is reachable on an access path of \(D'\). Second, note that relative access paths \(\pi'\) in \(D'\) starting at the conclusion of an instance \(i'\) of (FIX\text{letrec}) up to a marked assumption that is discharged at \(i'\) descend from a thread \(\pi\) in \(D\) from the conclusion of an application \(\iota\) of (FIX) up to a marked assumption that is discharged at \(\iota\). Since by Proposition 8.3 the thread \(\pi'\) passes at least one instance of a rule (\(\Lambda\)) or (\(\oplus\)), this is also the case for \(\pi\). As a consequence, all cycles on relative access paths are guarded. Thus \(D\) is guarded. Hence all occurrences of rule names (FIX\text{letrec}) in \(D'\) rightly label occurrences of this rule, and \(D'\) is a derivation in ann-Reg\text{letrec}, which moreover is guarded.

\[\text{Example 8.9.}\]

The closed derivation \(\hat{D}_1\) in Example 8.5 can be transformed into the following closed derivation in ann-Reg\text{letrec}:

\[
\begin{align*}
& ((\lambda x) \, c_u : M)^u \\
& \quad \Rightarrow (\lambda x) \, x : x^0 \\
& \quad \Rightarrow (\lambda x) \, x : M^x \\
& \quad \Rightarrow (\lambda x y) \, x y : M x^y \\
& \quad \Rightarrow (\lambda x y) \, c_u x y : M x^y \\
& \quad \Rightarrow (\lambda x y) \, \lambda y. c_u x y : M x^y \\
& \Rightarrow (\lambda x y) \, \lambda y. M x^y \\
& \Rightarrow \text{letrec } u = \lambda x y. u x y in u : M \\
& \text{FIX}_\text{letrec, } u
\end{align*}
\]

Now we concentrate on the remaining matter of proving that the \(\lambda\text{letrec}-\text{term}\) obtained by the annotation process from a closed derivation in Reg\text{letrec} to one in ann-Reg\text{letrec} does indeed unfold to the infinite \(\lambda\)-term it annotates. For this, we establish a proof-theoretic transformation from derivations in ann-Reg\text{letrec} to derivations in Reg\text{letrec}.

\[\text{Lemma 8.10 (from ann-Reg\text{letrec} to Reg\text{letrec}-derivations).}\]

Let \(D\) be a closed derivation in ann-Reg\text{letrec} with conclusion (\(\lambda x\)) \(L : M\). Then \(U(L)\downarrow\), and \(D\) can be transformed into a closed derivation \(D'\) in Reg\text{letrec} with conclusion (\(\lambda x\)) \(M\) by:

- replacing each formula occurrence \(o\) of (\(\lambda y\)) \(P : N\) in \(D\) by an occurrence of the formula \(U((\lambda y)\, \text{letrec } B \in \hat{P}) = (\lambda y)\) \(N\) in \(D'\), where \(B\) arises as the union of all
outermost binding groups in conclusions of instances of (FIX) at or below \( o \), and where \((\lambda \bar{y}) P = (\lambda \bar{y}) \tilde{P}\[ \tilde{f} := \bar{c}_f \] and \( \tilde{f} \) is comprised of the recursion variables occurring in \( B \) and \( \bar{c}_f \) are distinct constants for \( \tilde{f} \) as chosen by \( \mathcal{D} \); the unfoldings involved here are always defined.

**Proof.** Let \( \mathcal{D} \) be a closed derivation in \( \text{ann-Reg}^*_0 \) with conclusion \((\cdot)L: M\).

By Proposition 8.8, \( \mathcal{D} \) can be transformed into a closed derivation \( \mathcal{D}_1 \) in \( \text{ann-Reg}^*_{\text{letrec}} \) with the same conclusion. Due to Proposition 8.7, by dropping the infinite terms in \( \mathcal{D}_1 \), a derivation \( \mathcal{D}_2 \) in \( \text{Reg}^*_{\text{letrec}} \) with conclusion \((\cdot)L\) can be obtained. Then it follows from Theorem 6.24 that \( \mathcal{U}(L) \), that is, that \( L \) unfolds to an infinite \( \lambda \)-term.

We have to show that the transformation of \( \mathcal{D} \) into \( \mathcal{D}' \) as described in the statement of the lemma is, on the one hand, possible (that is, the unfolding of each prefixed \( \lambda_{\text{letrec}} \)-term is indeed defined), and on the other hand, that the prooftree \( \mathcal{D}' \) obtained by these replacements is indeed a valid derivation in \( \text{Reg}^*_0 \).

We argue for the possibility of these replacements and for their correctness locally, that is by carrying out the replacements from the bottom of \( \mathcal{D} \) upwards, thereby recognising for every replacement step that it is possible, and that it indeed produces a valid inference in \( \text{Reg}^*_0 \).

As a typical example of the arguments necessary to establish this fact, we consider a derivation \( \mathcal{D} \) in \( \text{ann-Reg}^*_0 \) with in it an instance of \((\lambda \bar{x} y)\) that immediately succeeds an instance of (FIX):

\[
\begin{align*}
D_0 &\quad (\lambda \bar{x} y)\ L_0 [u := c_u] : M_0 \\
&\quad (\lambda \bar{x} y)\ \text{letruc} u = L_0 \text{ in } u : M_0 \\
&\quad (\lambda \bar{x})\ (\lambda y.\ \text{letruc} u = L_0 \text{ in } u) : \lambda y. M_0 \\
D'_0 &\quad \lambda
\end{align*}
\]

According to the statement of the lemma, \( \mathcal{D} \) is transformed into the following \( \text{Reg}^*_0 \)-prooftree:

\[
\begin{align*}
\ldots &\quad \mathcal{U}((\lambda \bar{x} y)\ \text{letruc} B_0, u = \bar{L}_0, B' \text{ in } u) = (\lambda \bar{x} y) M_0 \ldots \\
D'_0 &\quad \mathcal{U}((\lambda \bar{x} y)\ \text{letruc} B_0, u = \bar{L}_0 \text{ in } \bar{L}_0) = (\lambda \bar{x} y) M_0 \\
&\quad \mathcal{U}((\lambda \bar{x} y)\ \text{letruc} B_0, u = \bar{L}_0 \text{ in } u) = (\lambda \bar{x} y) \text{ in } \bar{L}_0 \\
\mathcal{D}'_0 &\quad \mathcal{U}((\cdot)\ \text{letruc in } L) = (\cdot) M
\end{align*}
\]

where \( B_0 \) arises as the union of all outermost binding groups in conclusions of instances of (FIX) strictly below the visible instance of (FIX), and \( B' \) is the union of all outermost binding groups in conclusions of instances of (FIX) strictly above the visible instance of (FIX) and below the indicated marked assumptions (this binding group differs for different marked assumptions of this assumption class), and where \( \bar{L}_0 \) is the result of
replacing in \( L_0 \) all occurrences of constants \( c_f \) by the recursion variable \( f \) from which it originates.

Now assuming that the unfolding in the conclusion of the visible instance of (\( \oplus \)) has been shown to exist, we want to recognize that this instance and the instance of (FIX) above are valid instances in \( \text{Reg}_{\text{ann}}^\omega \). For the instance of (\( \lambda \)) we have to show:

\[
\mathcal{U}((\lambda x)\text{letrec } B_0 \text{ in } \lambda y.\text{letrec } u = \tilde{L}_0 \text{ in } u) \downarrow
\]

\[
\implies \text{for a term } \lambda \tilde{x}y. N_0: \quad \mathcal{U}((\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } \tilde{L}_0) \downarrow = (\lambda \tilde{x}y)N_0
\]

\[
\wedge \mathcal{U}((\lambda \tilde{x})\text{letrec } B_0 \text{ in } \lambda y.\text{letrec } u = \tilde{L}_0 \text{ in } u) = (\lambda \tilde{x})\lambda y. N_0
\]

This, however, is an easy consequence of the following \( \to_{\text{\( \gamma \)}} \)-rewrite steps:

\[
(\lambda \tilde{x})\text{letrec } B_0 \text{ in } \lambda y.\text{letrec } u = \tilde{L}_0 \text{ in } u \to_{\text{\( \gamma \)}} (\lambda \tilde{x})\lambda y. B_0 \text{ in letrec } u = \tilde{L}_0 \text{ in } u
\]

\[
\to_{\text{\( \gamma \),letrec}} (\lambda \tilde{x})\lambda y. B_0, u = \tilde{L}_0 \text{ in } u
\]

in view of the fact that, by Lemma 3.16, unfolding is unique normalising (in at most \( \omega \) steps). And for the instance of (FIX) we have to show:

\[
\mathcal{U}((\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } u) \downarrow \implies \mathcal{U}((\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } \tilde{L}_0) \downarrow = (*)
\]

\[
\wedge \mathcal{U}((\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0, B' \text{ in } u) \downarrow = (*)
\]

(ajaxually the statement as in the second line has to be shown for every binding-group \( B' \) that occurs for marked assumptions discharged at the instance of (FIX)). This implication is a consequence of the \( \to_{\text{\( \gamma \)}} \)-rewrite steps:

\[
(\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } u \to_{\text{\( \gamma \),rec}} (\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } \tilde{L}_0
\]

\[
(\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0, B' \text{ in } u \to_{\text{\( \gamma \),red}} (\lambda \tilde{x}y)\text{letrec } B_0, u = \tilde{L}_0 \text{ in } u
\]

again in view of the statement of Lemma 3.16.

The arguments used here are typical, and can be carried out similarly also for showing that axioms (\( \emptyset \)), and instances of rules (\( \oplus \)) and (S) in \( \text{ann-Reg}_{\text{ann}}^\omega \)-derivations give rise to, under the transformation described in the statement of the lemma, valid instances of axioms (\( \emptyset \)), and instances of (\( \oplus \)) and (S), respectively, in \( \text{Reg}_{\text{ann}}^\omega \)-derivations. \( \square \)

**Theorem 8.11.** If \( \to_{\text{ann-Reg}_{\text{ann}}^\omega } (L) \to M \) holds for a \( \lambda_{\text{letrec}} \)-term \( L \) and an infinite \( \lambda \)-term \( M \), then \( L \) unfolds to, and hence expresses, \( M \).

**Proof.** Suppose that \( D \) is a closed derivation in \( \text{ann-Reg}_{\text{ann}}^\omega \) with conclusion (\( \emptyset \)) \( L \to M \). Lemma 8.10 entails that \( L \) unfolds to an infinite \( \lambda \)-term, and moreover, that \( D \) can be transformed into a closed derivation \( D' \) in \( \text{Reg}_{\text{ann}}^\omega \) with conclusion (\( \emptyset \)) \( L \to M \). Then it follows by Theorem 6.17 (applying soundness of \( \text{Reg}_{\text{ann}}^\omega \) with respect to the property of \( \lambda_{\text{letrec}} \)-terms to unfold to an infinite \( \lambda \)-term that \( \mathcal{U}(L) = M \), and hence that \( L \to M \). In this way we have found a \( \lambda_{\text{letrec}} \)-term \( L \) that expresses \( M \). \( \square \)

We now arrive at our main characterisation result.

**Theorem 8.12.** An infinite \( \lambda \)-term is \( \lambda_{\text{letrec}} \)-expressible if and only if it is strongly regular.
Proof. Let \( M \) be an infinite \( \lambda \)-term. The direction \( "\Rightarrow" \) is the statement of Theorem 6.26. For showing the direction \( "\Leftarrow" \) in the statement of the theorem, we assume that \( M \) is strongly regular.

Then by Lemma 6.13, (ii), there exists a closed derivation \( D \) in \( \text{Reg}^* \) with conclusion \( ()M \). Due to Lemma 8.4, (i), \( D \) can be transformed into a derivation \( \hat{D} \) in \( \text{ann-Reg}_0^* \) with conclusion \( ()L : M \), for some \( \lambda_{\text{letrec}} \)-term \( L \). Then it follows by Theorem 8.11 that the \( \lambda_{\text{letrec}} \)-term \( L \) expresses \( M \).

As an immediate consequence of Theorem 8.12 and of Theorem 7.13 we obtain the following corollary, a summary of our main results.

**Corollary 8.13.** For all infinite \( \lambda \)-terms the following statements are equivalent:

(i) \( M \) is \( \lambda_{\text{letrec}} \)-expressible.

(ii) \( M \) is strongly regular.

(iii) \( M \) only contains finite binding–capturing chains.

9. \( \lambda \)-transition graphs

In this section we introduce the concept of \( \lambda \)-transition graphs. A \( \lambda \)-transition graph \( G \) of a term \( M \) can be understood as a nameless graphical representation closely related to the term graph of \( M \) in de-Bruijn notation. It is a graph that corresponds to the sub-ARS that is induced by \( M \) with respect to some \( \text{scope}^* \)-delimiting strategy for \( \text{Reg}^* \), but where no information can be extracted from the objects. Consider, for example the sub-ARSs displayed in Figure 8 and Figure 9 but ignore the prefixes by which the nodes are annotated. To capture the notion of ‘forgetting’ the term associated with each object of the ARS we use the formalism of labelled transition systems, in which only transitions are observable (see Section 2).

We will show a coinduction principle for infinite \( \lambda \)-terms: two \( \lambda \)-terms are equal if and only if they have bisimilar \( \lambda \)-transition graphs.

**Definition 9.1 (transition systems induced by CRSs).** Let \( A = (A, \Phi, \text{src}, \text{tgt}) \) be a sub-ARS, or a sub-ARS of a labelled version, of an ARS that is induced by a CRS \( C \) (see (Terese, 2003, 11.2.24) for a definition of induced ARSs). In particular, every step in \( \Phi \) carries information according to from which rule of \( C \) it stems from. By the LTS induced by \( A \) we mean the LTS \( L_A = (A, R, \Rightarrow) \) with transitions

\[
\Rightarrow := \{ (a, \rho, a') \mid (\exists \phi \in \Phi) \phi : a \Rightarrow a' \ \text{a step that stems from rule } \rho \}
\]

in which the steps in \( A \) according to rule \( \rho \) are interpreted as transitions with label \( \rho \). And for a subset \( R_0 \) of \( R \), by the LTS induced by \( A \) with silent \( R_0 \)-steps we mean the LTS \( L_{A,R_0} = (A, R, \Rightarrow') \) with

\[
\Rightarrow' := \{ (a, \rho, a') \mid (\exists \phi \in \Phi) \phi : a \Rightarrow R_0 : \rho : a' \ \text{where } \Rightarrow R_0 \ \text{are steps w.r.t. rules in } R_0, \ \text{and } \Rightarrow \rho \ \text{is a step w.r.t. rule } \rho \in R \setminus R_0 \}
\]

in which the steps in \( A \) according to rules in \( R_0 \) are interpreted as silent transitions, and the remaining rules as transitions according to their name.
Definition 9.2 (transition graph of an object). If $a$ is an object of the ARS $A$ that is induced by a CRS $C$ with rules $R$, and $(A, R, \rightarrow) = L(a \rightarrow)$ the LTS induced by $(a \rightarrow)$, then we call $G_A(a) = (S, A, a, \rightarrow)$ the transition graph of $a$.

For an LTS $L(a \rightarrow), R_0 = (A, R, \rightarrow)$ with silent $R_0$-steps, we call $G_{A,R_0}(a) = (S, A, a, \rightarrow)$ the transition graph of $a$ with silent $R_0$-steps.

Definition 9.3 ($\lambda$-transition graph). We call a labelled transition graph $G = (S, A, x, \rightarrow)$ a $\lambda$-transition graph if:

- it is connected
- the labels are $A = \{\lambda, S, @_0, @_1\}$
- there are no infinite paths in $G$ consisting solely of $S$-transitions

every state belongs to one of the following kinds: $\lambda$-states, $S$-states, and $@$-states, where

- a $\lambda$-state $s$ is the source of precisely one $\lambda$-transition, and no other transitions:
  $\{(l, t) \mid (s, l, t) \in \rightarrow\} = \{\{\lambda, t\}\}$ for some $t \in S$.

- a $S$-state $s$ is the source of precisely one $S$-transition, and no other transitions:
  $\{(l, t) \mid (s, l, t) \in \rightarrow\} = \{\{S, t\}\}$ for some $t \in S$.

- a $@$-state $s$ is the source of precisely one $@_0$-transition and one $@_1$-transition, but no other transitions:
  $\{(l, t) \mid (s, l, t) \in \rightarrow\} = \{\{@_0, t\}, \{@_1, u\}\}$ for some $t, u \in S$.

Proposition 9.4. Let $S^+$ be a scope$^+$-delimiting strategy of $\text{Reg}^+$. For every term $M \in \text{Ter}(\lambda^\omega)$ the transition graph $G_{S^+}(M)$ of $M$ is a $\lambda$-transition graph.

**Proof.** In transition graphs $G_{S^+}(M)$, infinitely many successive $S$-transitions are not possible because in the ARS that induces $G_{S^+}(M)$, the rewrite relation $S$ is terminating, due to Proposition 4.8, (iv).

Along the lines of Proposition 9.4 we can also view transition graphs of $\text{letrec}$-terms as $\lambda$-transition graphs, but only when treating unfolding steps as silent transitions. As hinted before in Remark 5.7, here the restriction of scope-delimiting (and scope$^+$-delimiting) strategies to ones that prevent indeterminism in the application of unfolding rules is relevant.

Proposition 9.5. Let $S^+$ be a scope$^+$-delimiting strategy of $\text{Reg}_{\text{letrec}}^+$. For every term $L \in \text{Ter}(\lambda_{\text{letrec}}^\omega)$, the transition graph $G_{S^+}(L)$ of $L$ is a $\lambda$-transition graph.

Definition 9.6 ($\lambda$-transition graph of a term).

(i) Let $M \in \text{Ter}(\lambda^\omega)$. For a scope$^+$-delimiting strategy $S^+$ of $\text{Reg}_{\text{letrec}}^+$ we call the transition graph $G_{S^+}(M)$ the $\lambda$-transition graph of $L$ with respect to $S^+$. And more generally, by a $\lambda$-transition graph of $M$ we mean a transition graph that is bisimilar to the transition graph of $M$ with respect to a scope$^+$-delimiting strategy $S^+$.

(ii) Let $L \in \text{Ter}(\lambda_{\text{letrec}})$ be a (prefixed) $S^+$-productive $\lambda_{\text{letrec}}$-term. For a scope$^+$-delimiting strategy $S^+$ of $\text{Reg}_{\text{letrec}}^+$ such that $L$ is $S^+$-productive, we call the transition graph $G_{S^+}(L)$ the $\lambda$-transition graph of $L$ with respect to $S^+$. And more generally, by a $\lambda$-transition graph of $L$ that is bisimilar to the transition
graph of $L$ with respect to a scope$^+$-delimiting strategy $S^+$ with the property that $L$ is $S^+$-productive.

For prefixed $\lambda$-terms in $\text{Ter}(\lambda^\infty)$ and in $\text{Ter}(\lambda_{\text{letrec}})$ we use the terms ‘$\lambda$-transition graph’ and ‘transition graph’ synonymously. We also speak of $\lambda$-transition graphs of terms $L \in \text{Ter}(\lambda_{\text{letrec}})$ or $M \in \text{Ter}(\lambda^\infty)$ by which we refer to the $\lambda$-transition graphs of $L$ and $M$, respectively.

**Theorem 9.7 (coinduction principle for $\lambda^\infty$).** For all infinite $\lambda$-terms $M$ and $N$ the following statements are equivalent:

(i) $M = N$.
(ii) $\xrightarrow{\text{EQ}^\infty} M = N$.
(iii) $M$ and $N$ have bisimilar $\lambda$-transition graphs.

**Proof.** In view of Proposition 6.15, (ii), the logical equivalence between (i) and (ii), and the fact that (i) $\Rightarrow$ (iii) clearly holds, it suffices to show that (iii) $\Rightarrow$ (ii) holds.

For this, suppose that $G_{S^1}(M)$ and $G_{S^2}(N)$ are bisimilar for some scope$^+$-delimiting strategies $S^1$ and $S^2$ for $\text{Reg}$. But now bisimilarity of these transition graphs guarantees that a derivation $D$ in $\text{EQ}^\infty$ with conclusion $()M = ()N$ can be constructed such that all threads in $D$ correspond to $\xrightarrow{S^1}$-rewrite sequences on $M$ and to $\xrightarrow{S^2}$-rewrite sequences on $M$, respectively. If the construction process is organised in a depth-fair manner (for example, all non-axiom leafs at depth $n$ are extended by appropriate rule instances, before extensions are carried out at depth greater than $n$), then in the limit a completed derivation $D^\infty$ with conclusion $()M = ()N$ is obtained. This establishes $\xrightarrow{\text{EQ}^\infty} M = N$. $\square$

**Conjecture 9.8 (coinduction principle for $\lambda_{\text{letrec}}$).** For all $L_1, L_2 \in \text{Ter}(\lambda_{\text{letrec}})$ it holds that $L_1 = L_2$ if and only if $L_1$ and $L_2$ have bisimilar $\lambda$-transition graphs.

**Remark 9.9 (only $\text{Reg}^*$ defines nameless representations).** Note that the coinduction principle does not hold if the definition of a ‘$\lambda$-transition graph of a term’ (Definition 9.6) is specified in terms of $\text{Reg}$ instead of $\text{Reg}^*$.

As an example consider the $\text{Seag}$ sub-ARS in Figure 7 whose corresponding LTS would be obtained from four different $\lambda$-terms:

\[
\lambda xy.xx\quad \lambda xy.xy\quad \lambda xy.yy\quad \lambda xy.yyy
\]

The understanding of $\lambda$-transition graphs as nameless representations of an infinite $\lambda$-terms implies that from such a graph the corresponding $\lambda$-term can be extracted. We define a function for this purpose by means of a CRS which implements the assembly of a $\lambda$-term from the infinite unfolding of a $\lambda$-transition graph. The function is closely related to the $\text{Parse}^*$ in the sense that $\text{Parse}^*$ does both destruct and reconstruct the scrutinised term while $\text{readback}$ only implements the reconstruction.

**Definition 9.10 (readback).**

\[
\text{readback} : \text{Ter}^\infty(\{0, \lambda, @, S\}) \rightarrow \text{Ter}(\lambda^\infty)
\]

\[
t \mapsto \text{readback}(t) := \text{infinite normal form of } \text{rw}_0(t)
\]

w.r.t. the following CRS:
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\[ \text{rw}_n(X_1, \ldots, X_n, \lambda(t_0)) \rightarrow \text{abs}([x] \text{rw}_{n+1}(X_1, \ldots, X_n, x, t_0)) \]

\[ \text{rw}_n(\lambda(t_0, t_1)) \rightarrow \text{app}(\text{abs}([x] \text{rw}_n(\lambda(t), t_0)), \text{abs}([x] \text{rw}_n(\lambda(t), t_1))) \]

\[ \text{rw}_{n+1}(\lambda(x, S(t_0))) \rightarrow \text{rw}_n(\lambda(t_0)) \]

\[ \text{rw}_n(X_1, \ldots, X_n, 0) \rightarrow X_n \]

The function is partial because \( \text{rw}_n \) is unproductive for infinite S-chains. That restriction comes forth accordingly in the definition of \( \lambda \)-transition graphs (Definition 9.3). The function is thus complete on the subset of \( \text{Ter}^\infty([0, \lambda, \@, S]) \) that is obtained from unfolding a \( \lambda \)-transition graph.

10. Conclusion and Outlook

In this work we have introduced a number of formalisms for relating infinite \( \lambda \)-terms and finite terms in the \( \lambda \)-calculus with letrec to each other. In the following we recapitulate the most important concepts briefly.

We provide CRS signatures to define the set of infinite \( \lambda \)-terms and the set of \( \lambda_{\text{letrec}} \)-terms, which we connect by the CRS \( R_\square \) for unfolding \( \lambda_{\text{letrec}} \)-terms to their corresponding \( \lambda \)-term. To determine which \( \lambda_{\text{letrec}} \)-terms have an infinite unfolding we identify productive \( \lambda_{\text{letrec}} \)-terms.

To characterise the set of \( \lambda \)-terms for which there exists a corresponding \( \lambda_{\text{letrec}} \)-term (such that the former can be obtained from the latter via unfolding) we establish a framework of formalisms for ‘observing’ \( \lambda \)-terms coinductively. Firstly we introduce prefixed \( \lambda \)-terms that enrich \( \lambda \)-terms by an abstraction prefix. On the prefixed terms we define the CRS \( \text{Reg}^+ \) in which a rewrite sequence corresponds to a deconstruction of a term along one of its paths. In that sense a prefixed term \((\lambda \overline{x})M\) can be understood as a ‘suspended decomposition’ which has not advanced into subterm \( M \) yet. Such a decomposition describes a path through the term by observations of the form \( \rightarrow_{\lambda}, \rightarrow_{\text{a}_0}, \rightarrow_{\@_1}, \rightarrow_{\text{del}}, \) where the latter delimits the scope* of an abstraction.

Since there is some freedom as to where scope*-delimiters can be placed, we define scope*-delimiting strategies to formalise specific possible choices eliminating that freedom and thereby making the observations deterministic except for the forking into the left or the right subterm of an application. By means of scope*-delimiting strategies we can formulate two important concepts: strong regularity and \( \lambda \)-transition graphs.

The intuitive understanding of strong regularity is the property of an infinite \( \lambda \)-term \( M \) that from \( M \) every ‘sufficiently eager’ scope*-delimiting strategy can only generate a finite number of terms. We then show that \( \lambda_{\text{letrec}} \)-expressibility coincides with strong regularity.

Every scope*-delimiting strategy defines a \( \lambda \)-transition graph of a term which can be viewed as a nameless graphical representation very similar to its term graph in de-Bruijn notation with the difference that \( S \)-nodes are not restricted to occur only near leafs but can be shared by variables. The eager scope*-delimiting strategy yields finite \( \lambda \)-transition graphs for strongly regular \( \lambda \)-terms.

We adapt the concepts of the CRS for observing terms, scope*-delimiting strategies, and \( \lambda \)-transition graphs and apply them to \( \lambda_{\text{letrec}} \) proving similar results as for \( \lambda^\infty \).
We provide a proof system that is sound and complete for the notion of strong regularity and which admits finite proofs for strongly regular \( \lambda \)-terms. We define an annotated version of the proof system which not unlike an attribute-grammar definition implements the extraction of a \( \lambda_{\text{letrec}} \)-term \( L \) from a proof for term \( M \) in that system, such that \( L \) unfolds to the \( M \). We show that every scope\(^*\)-delimiting strategy induces a proof and that from a proof a corresponding history-aware strategy can be deduced, which suggests a similar correspondence between \( \lambda \)-transition graphs and proofs.

The following results are within reach but not worked out yet:

— coinduction principle for \( \lambda_{\text{letrec}} \): For all \( L, P \in \text{Ter}(\lambda_{\text{letrec}}) \) it holds that \( L = P \) if and only if \( L \) and \( P \) have bisimilar \( \lambda \)-transition graphs.
— a proof system for unfolding equivalence of \( \lambda_{\text{letrec}} \)-terms
— a thorough coinductive treatment of \( \lambda \)-transition graphs and finality results
— finite representations of regular \( \lambda \)-terms as higher-order recursive program schemes (cf. Example 4.25) and their extractions from formalised proofs of regularity
— characterisation of \( \lambda_{\text{letrec}} \)-expressible preterms of infinite \( \lambda \)-terms as those that can be generated, up to \( \alpha \)-equivalence, by first-order recursive program schemes
— a terminating readback function to extract \( \lambda_{\text{letrec}} \)-terms from transition graphs

We feel that in gathering these results we have gained a new perspective on the \( \lambda \)-calculus with \( \text{letrec} \). We expect be able to use the concepts and formalisms introduced here to obtain related results of the following kinds:

— efficient decision of unfolding equivalence of \( \lambda_{\text{letrec}} \)-terms by means of the DFA-equivalence algorithm of Hopcroft and Karp (Hopcroft and Karp, 1971).
— a partial order for the amount of subterm sharing in a \( \lambda_{\text{letrec}} \)-term leading to
  — a notion of maximal sharing for \( \lambda_{\text{letrec}} \)
  — an efficient mechanism to derive the maximally shared form of a \( \lambda_{\text{letrec}} \)-term which generalises common subexpression elimination
— a higher-order rewriting framework founded on \( \lambda_{\text{letrec}} \) instead of the \( \lambda \)-calculus

References

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